Proceedings of ICRC 2001: 1856 © Copernicus Gesellschaft 2001

ICRC 2001

Calculation of path-length distribution for galactic cosmic rays using the stochastic differential equation technique

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Abstract. Cosmic ray propagation in a galactic diffusion model is considered. The equation for the path-length distribution function is presented in the form of a stochastic differential equation solved starting from an observer location, and the distribution of cosmic rays on matter thickness traversed in the interstellar gas is calculated. A realistic spatial distribution of cosmic ray sources and interstellar gas density are taken into account in the calculations. The obtained path-length distribution is close to the exponential one. This justifies the feasibility of the leaky box approximation for studies of interstellar transport and nuclear fragmentation of energetic stable nuclei. The clumpiness of the interstellar gas in the form of dense clouds and a correlation of gas density and cosmic-ray source density may lead to the deviation of the distribution function from an exponental at small path lengths.

1 Introduction

The study of the transport of energetic nuclei in the Galaxy requires a consideration of their spallation which is due to the interaction with the interstellar gas nuclei. Hundreds of isotopes should be included in the calculations. There is a powerful method to solve a set of transport equations for the generations of nuclei linked by nuclear fragmentation of parent isotopes into more light progenitors. This method, the weighted slab technique, consists of splitting of the problem into astrophysical and nuclear parts, see Berezinski *et al.* (1990). The nuclear fragmentation problem is solved in terms of the slab model wherein the cosmic ray beam is allowed to traverse a thickness, x g/cm^2, of the interstellar gas and these solutions are integrated over all values of x weighted with a distribution function $G(x)$ that is derived from an astrophysical propagation model. The standard weighted slab method (Protheroe *et al.* , 1981; Garcia-Muñoz *et al.* , 1987) gives only an approximate solution at low energies where nuclear cross sections and diffusion depend on energy, and ionization energy losses are significant. It also gives wrong results when the A/Z ratios are different for parent and daughter nuclei. A modification of the weighted slab method (Ptuskin *et al.* , 1996) makes it rigorous for the case of separable dependence of the diffusion coefficient on particle energy (or rigidity) and position with no convective transport. (The convective transport can be treated rigorously by this method in some special cases.)

Interpretation of data on cosmic ray composition and, in particular, on the abundance of stable secondary isotopes suggests an exponential form of the path-length distribution $G(x) \sim \exp(-x/X_e)$ with an energy dependent parameter X_e, the escape length. A position independent $X_e(E)$ is characteristic of the leaky box model, the approximation commonly used in practical calculations of cosmic ray propagation. The description of cosmic ray propagation in the diffusion model, that is closer to physical reality, is more complicated. However, at least in the case of stable and not very heavy nuclei, the solution of the diffusion equation for an observer at the galactic disk can be well approximated by the equation typical for the leaky box model. The relation between parameters of the flat-halo diffusion model and the leaky box model is given by the equation

$$X_e \approx \mu\beta cH/2D \quad (1)$$

valid at $\sigma \ll mh/(X_eH)$. Here βc is the particle velocity, μ is the surface mass density of the galactic disk, H is the height of the cosmic ray halo, D is the cosmic ray diffusion coefficient, σ is the total fragmentation cross section, m is the average mass of an interstellar atom, h is the height of the galactic gas disk, and it is assumed that $h \ll H$.

Investigations of path-length distributions in diffusion models were mainly made with the use of analytical methods (Berezinski *et al.* , 1990; Ptuskin *et al.* , 1997) and thus were limited by the models with simplified cosmic ray source and interstellar gas distributions that allow one to obtain analytic solutions. The Monte-Carlo simulations performed by Web-

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ber (1993) were also made for a simple case of $\delta(z)$ distribution of sources and an exponential distribution of gas in one dimension perpendicular to the galactic plane. Such simplified models are not acceptable for many problems of cosmic rays astrophysics. The analysis of diffuse galactic gamma-ray radiation generated by cosmic ray particles (Strong *et al.* , 1998) and the determination of cosmic ray age from the abundance of radioactive isotopes (Ptuskin & Soutoul , 1998) present examples of more advanced models.

In the present work we describe a prime numerical method of determination of path length distribution function in diffusion models with arbitrary three-dimensional distributions of cosmic ray sources, interstellar gas density, and arbitrary spatial dependence of the cosmic ray diffusion coefficient. This economic approach is based on the solution of a backward stochastic differential equations starting from an observer position. The objective is to study models that reflect realistic astrophysical conditions in the Galaxy and to find justification and limits for popular empirical models of cosmic ray propagation.

2 Path length distribution formalism

It was shown in Ptuskin *et al.* (1996) that one can present the cosmic ray intensity as

$$I_i(\mathbf{r},E) = \int_0^\infty dx G(x,\mathbf{r}) F_i(x,E). \quad (2)$$

where $F_i(x,E)$ represents the fragmentation and energy change of a particle of the ith species as a function of grammage, x, and energy, E, and $G(x,\boldsymbol{r})$is the path length distribution of a characteristic particle species. G obeys the following diffusion equation that depends on the interstellar gas distribution and on the properties of particle wandering in the Galaxy but does not depend on particle energy or on the type of nucleus:

$$\rho(\mathbf{r})\frac{\partial G}{\partial x} - \frac{1}{3}\nabla l(\mathbf{r})\nabla G = s(\mathbf{r})\delta(x), \quad (3)$$

where $s(\mathbf{r})$ is the spatial distribution of cosmic ray sources. We shall assume the free exit of particles at the galactic cosmic ray halo boundaries, i.e. $G\mid_\Sigma = 0$.

3 Stochastic differential equation technique

Using the reciprocity principle for the corresponding Green functions (Morse & Feshbach , 1953), the solution of equation (3) at the observer position $\mathbf{r}_0$ can be presented as

$$G(\mathbf{r}_0,x) = \int\int\int d^3r s(\mathbf{r})\varphi(\mathbf{r},x), \quad (4)$$

where the Green function φ obeys the equation

$$\rho(\mathbf{r})\frac{\partial\varphi}{\partial x} - \nabla\frac{l(\mathbf{r})}{3}\nabla\varphi = \delta^3(\mathbf{r}-\mathbf{r}_0)\delta(x), \varphi\mid_\Sigma = 0. \quad (5)$$

Introducing a new function $\psi = \rho(\mathbf{r})\varphi$ (we assume here that density $\rho(\mathbf{r}) \neq 0$ everywhere in the system including its boundaries), one can obtain the following equation for ψ in a canonical Fokker-Planck form:

$$\frac{\partial\psi}{\partial x} - \nabla^2\left(\frac{l}{3\rho}\psi\right) + \nabla\left(\frac{(\nabla l)}{3\rho}\psi\right) = \delta^3(\mathbf{r}-\mathbf{r}_0)\delta(x). \quad (6)$$

This leads to the following stochastic differential equation for the trajectory $\mathbf{r}(x)$ of diffusing particle (Gardiner , 1983):

$$dr_i = \frac{(\nabla l)}{3\rho}dx + \sqrt{\frac{2l}{3\rho}}dW_i \quad (7)$$

with the initial condition $\mathbf{r} = \mathbf{r}_0$ at $x = 0$. Here dW_j is a Wiener process given by the Gaussian distribution $P(dW_j) = (2\pi dx)^{-1/2}\exp\left(-dW_j^2/2dx\right)$. The absorption of particles is assumed at the boundaries of the system.

Eq. (6) is the equation for the distribution function ψ. Assuming that a large number N of random particle trajectories $\mathbf{r}^{(m)}(y)$ were emitted from the point $\mathbf{r}_0$, one can present the function ψ as

$$\psi = \frac{1}{N}\sum_{m=1}^{N}\delta(\mathbf{r}-\mathbf{r}^{(m)}(x))\ \mid_{\mathbf{r}\in V}; \quad \psi = 0\ \mid_{\mathbf{r}\notin V}. \quad (8)$$

Eqs. (4), (8) now give the following expression for the distribution function G:

$$G(\mathbf{r}_0,x) = \frac{1}{N}\sum_{m=1}^{N}\frac{s(\mathbf{r}^{(m)}(x))}{\rho(\mathbf{r}^{(m)}(x))} = \left\langle\frac{s(\mathbf{r}(x))}{\rho(\mathbf{r}(x))}\right\rangle. \quad (9)$$

The condition of particle absorption at the outer halo boundaries implies that $s(\mathbf{r}^{(m)}(x)) = 0$ at $x > x_a$, where x_a is the matter thickness traversed by a particle up to the moment it reaches the boundary.

Eqs. (7) and (9 determine the path-length distribution function G through a family of diffusion trajectories $\mathbf{r}^{(m)}(x)$, $m = 1, 2...N$, $N >> 1$ emitted from the observer position $\mathbf{r}_0$.

4 Numerical simulations

The numerical calculations of the path-length distribution function can be fulfilled with the use of Eqs. (7), (9). We illustrate with two examples.

Let us consider the flat halo model where cosmic ray sources have the distribution $s \sim \exp(-\mid z\mid/h_s)$ with the characteristic scale height $h_s = 200$ pc. The total thickness of the galactic cosmic ray halo is taken to be $2H = 5.7$ kpc. The hydrogen gas distribution consists of 4 layers with exponential profiles: $n_H(z) = \sum n_i exp(-\mid z\mid/h_i)$, $i = 1,2,3,4$. The component with parameters $n_1 = 0.45$ cm^{-3}, $h_1 = 130$ pc represents the smeared out contribution of small numerous neutral clouds, the component $n_2 = 0.21$ cm^{-3}, $h_2 = 200$ pc represents the more extended warm medium, the component $n_3 = 0.025$ cm^{-3}, $h_3 = 1$ kpc represents the ionized hot gas, and the component $n_4 = 0.44$ cm^{-3}, $h_4 = 60$ pc represents the molecular hydrogen. The interstellar medium consists of 90% of hydrogen and 10% of helium atoms. The interstellar helium is distributed in the same way as hydrogen. The accepted diffusion mean free path is $l = 1$ pc, that corresponds to the diffusion coefficient $D = 3\times10^{28}\beta$ cm^2/s typical for cosmic rays with energies about few GeV/nucleon (at $a = 1$).

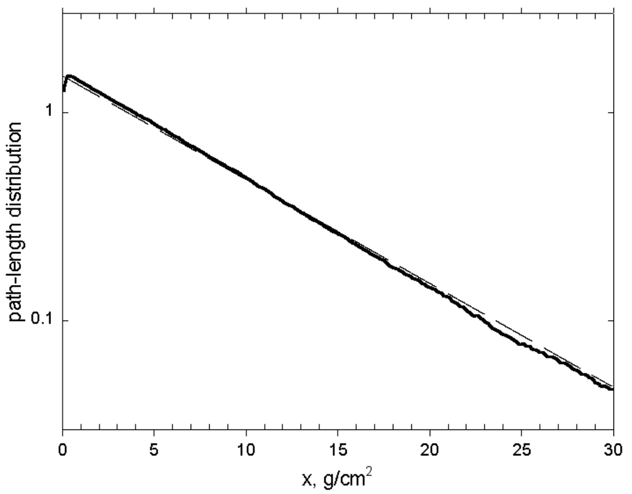

Fig. 1. Path-length distribution function in a flat halo diffusion model with a few layer gas distribution described in the main text (solid line). The dashed line shows an exponential distribution.

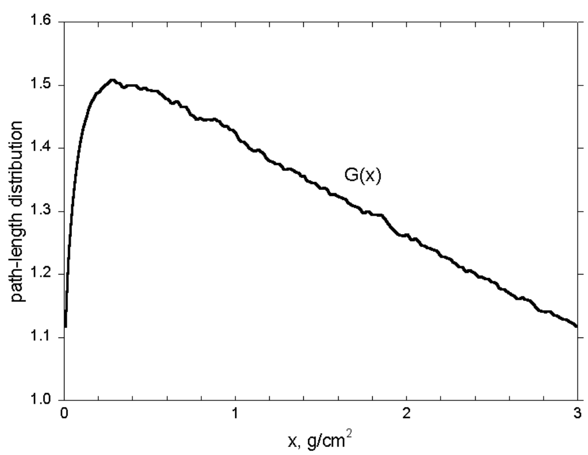

Fig. 2. Detailed structure at small path lengths of the same distribution function as in Figure 1.

Figures 1 and 2 show the result of calculations of path length distribution $G(x)$ based on 20000 random particle trajectories released at an observer position at the galactic midplane. The distribution is normalized to $G(0) = 1$. The exponential distribution with the escape length $X_e = 8.7$ g/cm^2 obtained with the use of equation (1) is shown for comparison. It is evident that the actual distribution is very close to the exponential one except for small path lengths $x < x_m \sim 0.5$ g/cm^2. This explains the practicality of the leaky box model for the interpretation of data on stable nuclei in cosmic rays.

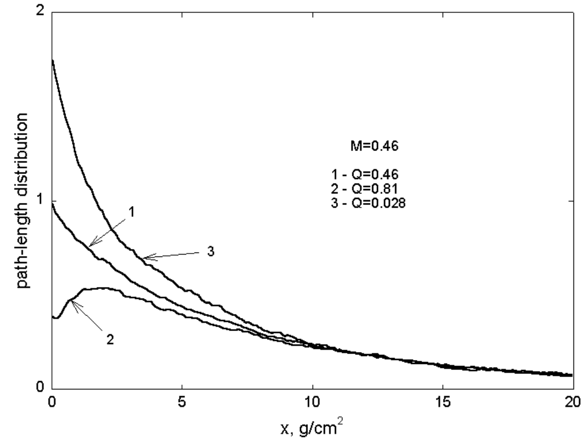

Fig. 3. Distribution on path length in cloudy interstellar medium for different values of the total fraction Q of cosmic ray sources inside the clouds at fixed fraction M of the total mass of gas contained in the clouds.

More considerable deviation from the exponential distribution is possible in the models where the clumpiness of interstellar gas in the form of dense clouds is taken into account and where the distribution of cosmic ray sources is correlated with the gas distribution (Ptuskin & Soutoul , 1990; Ptuskin *et al.* , 1997; Cowsik & Wilson , 1973, the last being the seminal work in this area). This effect is demonstrated in Figure 3. The path length distribution was calculated in the flat halo model with cosmic ray halo thickness $2H = 2$ kpc. The two-component distributions were used for the interstellar gas and the cosmic ray sources. The uniform component with constant density occupied the intercloud space in the galactic disk of total thickness $2h = 200$ pc. The multiple cylindrical clouds of radius $R_{cl} = 30$ pc lined up in the galactic disk at distances $d = 500$ pc from each other. The gas density was chosen to be $n_H = 1$ cm^{-3} in the intercloud space and $n_{H,cl} = 30$ cm^{-3} inside the clouds whereas the corresponding source densities were varied. The accepted diffusion mean free pass was $l = 0.3$ pc. The three curves in Figure 3 correspond to the cases when the fraction Q of all cosmic ray sources contained in the clouds was correspondently equal, considerably more, and considerably less than the fraction M of the total interstellar gas mass contained in the clouds. The exponential distribution $G(x)$ was obtained for $Q = M$. The deficit at small path lengths (the truncation) was observed for $Q > M$, and the excess at small path lengths was observed for $Q < M$.

These results are in full agreement with previous analytic work on the theory of cosmic ray diffusion and nuclear fragmentation in strongly inhomogeneous interstellar medium. The possible existence of truncation in the exponential path length distribution for galactic cosmic rays was discussed over a long period of time (Shapiro & Silberberg , 1970;

Garcia-Muñoz *et al.* , 1987; Soutoul *et al.* , 1990) . The discussion concerns, in particular, the relative abundance of secondary boron and sub iron elements. The increase of about 10 percent of sub iron secondaries may exist in galactic cosmic rays compared with the prediction based on the pure exponential path-length distribution function with the escape length determined from the fit to the observed abundance of boron. The truncation may occur at about $(0.2...0.3)X_e$. The spallation of cosmic ray nuclei with large cross sections is most sensitive to the behaviour of the path-length distribution at small x. For example, a relativistic Pb nucleus has the desruction length as short as 1 g/cm^2 (Waddington , 1996).

5 Conclusion

The correspondence between the Fokker-Planck equation for particle distribution function and the stochastic differential equation, the Ito equation, for random particle trajectories was proven to be useful for the solution of various problems in mathematics, physics, and economics. It was employed in particular for studies of energetic particle acceleration and transport in space plasmas, see Krulls & Achterberg (1994); Fichtner *et al.* (1996); Zhang (1999, 2000).

In the present paper, it is shown that the technique of stochastic differential equations is well adoptable to the calculation of cosmic-ray path-length distribution in the Galaxy. The remarkably simple Eq. (9) allows one to find the distribution function $G(x)$ by calculating random trajectories of relativistic particles backward from the observer location. The presented calculations with a realistic, many component, spatial distribution of interstellar gas taken into account showed that the path-length distribution is close to the exponential one. This confirms the old theoretical result (Berezinski *et al.* , 1990) that the disk-halo diffusion model is close to the leaky box model for the interpretation of data on abundances of stable nuclei in cosmic rays. The calculations where the presence of dense gas clouds was taken into account demonstrated that a significant deviation from the exponential distribution is possible at small path-lengths in the case of a strong correlation of gas density and cosmic ray source density inside the clouds. This is also in agreement with analytic theory predictions.

The case with dense clouds refers to the long-discussed question about possible truncation of exponential path-length distribution. The experiments like the proposed HNX Mission (Binns *et al.* , 2001), where the elemental abundance of cosmic ray nuclei up to actinides can be measured, would be very informative for the determination of the exact form of cosmic ray distribution function on path-length and for the investigation of possible gas density - cosmic-ray source density correlation.

Acknowledgements. This work was supported by NASA grant NAG5-7069. The VP work was supported by RFBR-01-02-17460 grant.

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