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Analytical solution of 3D cosmic-ray diffusion in boundaryless halo (II) – two component model –

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Abstract.

We derive an analytical solution of three dimensional cosmic-ray diffusion in the Galaxy with two-component scale heights, one corresponding to the disk and the other to the halo, assuming that three critical parameters, D: diffusion coefficient, n: gas density, and s: cosmic-ray source density, are of the exponential type in both r (radial distance from the disk center) and z (vertical distance from the galactic plane), i.e., D, n and s have two scale heights, one set with $[z_{gD}, z_{gn}, z_{gs}]$ for $z \leq z_c$ and the other with $[z_{hD},\, z_{hn},\, z_{hs}]$ for $z \geq z_c$. We expect the former three heights are of the order of magnitude with a few hundreds pc, while the latter three are of the order of magnitude more than a few kpc, much larger than the former ones.

1 Introduction

In another paper presented in this volume (Shibata 2001, hereafter named paper I), we gave an analytical solution for the cosmic-ray (CR) diffusion, assuming that the distribution shapes of three critical parameters, D: diffusion coefficient, n: gas density, and s: CR source density, are of the exponential type in both r (radial distance from the disk center) and z (vertical distance from the galactic plane),

$$D(r,z) \;=\; D_0 \exp[r/r_D + |z|/z_D], \qquad (1a)$$

$$n(r,z) = n_0 \exp[-(r/r_n + |z|/z_n)], \qquad (1b)$$

$$s(r,z) \;=\; s_0 \exp[-(r/r_s + |z|/z_s)]. \qquad (1c)$$

Practically, however, it is quite likely that the above three parameters have two components, one corresponding to the disk and another to the halo. So, introducing a critical distance z_c from the galactic plane, we assume

$$[z_D, z_n, z_s] \;=\; \begin{cases} [z_{gD}, z_{gn}, z_{gs}], & \text{for } |z| \leq z_c \quad (2a) \\ [z_{hD}, z_{hn}, z_{hs}], & \text{for } |z| \geq z_c \quad (2b) \end{cases}$$

$$[D_0, n_0, s_0] \;=\; \begin{cases} [D_{g0}, n_{g0}, s_{g0}], & \text{for } |z| \leq z_c \quad (3a) \\ [D_{h0}, n_{h0}, s_{h0}], & \text{for } |z| \geq z_c \quad (3b) \end{cases}$$

In Eq. $(2a)$, three parameters, $[z_{gD}, z_{gn}, z_{gs}]$, correspond to the scale heights of D, n and s in the disk, while $[z_{hD}, z_{hn}, z_{hs}]$ in Eq. $(2b)$ to those in the halo. Naturally, we expect

$$z_{hD} \gg z_{gD}, \quad z_{hn} \gg z_{gn}, \quad z_{hs} \gg z_{gs},$$

and probably z_{hD}, z_{hn} and z_{hs} might be one order of magnitude larger than z_{gD}, z_{gn} and z_{gs}.

Taking the continuation condition at $z = z_c$ into account, we have following three constraints for these parameters,

$$D_{g0} \exp[z_c/z_{gD}] = \; D_{h0} \exp[z_c/z_{hD}], \qquad (4a)$$

$$n_{g0} \exp[-z_c/z_{gn}] = n_{h0} \exp[-z_c/z_{hn}], \qquad (4b)$$

$$s_{g0} \exp[-z_c/z_{gs}] = s_{h0} \exp[-z_c/z_{hs}]. \qquad (4c)$$

In the present paper, we derive an analytical solution of 3D CR diffusion in the case of two components in scale heights for the diffusion coefficient, gas density and the CR source density.

Throughout this paper, we distinguish all variables related to the disk ($z \leq z_c$) from those to the halo ($z \geq z_c$) by attaching the subscripts, "g" and "h", to them, respectively, and we use the same notations as those used in paper I.

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2 Solution of the diffusion equation

2.1 Fundamental solution

The diffusion equation and the boundary condition are presented in paper I, and the difference here from it is only the continuation condition at $z = z_c$. So, the fundamental solution of the diffusion equation is completely the same as in the case of the one component model (see Eq. (14) in paper I),

$$\varphi_k(u; r_0, u_0) = aA_k(\Lambda) + bB_k(\Lambda), \tag{5}$$

$$\Lambda(u) = \lambda_0 U(u) = \lambda_0 \mathrm{e}^{-u}, \tag{6a}$$

$$\text{with} \quad u = r/\bar{r} + |z|/\bar{z}, \tag{6b}$$

where we introduced following functions and variable,

$$A_k(\Lambda) = \Lambda^\nu I_{\nu_k}(\Lambda), \quad B_k(\Lambda) = \Lambda^\nu K_{\nu_k}(\Lambda), \tag{7}$$

$$\lambda_0 = \sqrt{\frac{n_0 v \sigma}{D_0}}\,\bar{z}. \tag{8}$$

Before going to the details of the procedure in the derivation of the solution, we summarize variables,

$$\Lambda_i \equiv \lambda_{i0} U(u_i) \quad \text{with} \quad u_i = r/\bar{r} + |z|/\bar{z}_i, \tag{9a}$$

$$\Lambda_{ic} \equiv \lambda_{i0} U(u_{ic}) \quad \text{with} \quad u_{ic} = r/\bar{r} + |z_c|/\bar{z}_i, \tag{9b}$$

$$\Lambda_{i0} \equiv \lambda_{i0} U(u_{i0}) \quad \text{with} \quad u_{i0} = r_0/\bar{r} + |z_0|/\bar{z}_i, \tag{9c}$$

where i denotes "g" (disk) or "h" (halo), and

$$\lambda_{i0} = \sqrt{\frac{n_{i0} v \sigma}{D_{i0}}}\,\bar{z}_i, \tag{8$'$}$$

$$\text{with} \quad \frac{1}{\bar{z}_i} \;=\; \frac{1}{2}\left(\frac{1}{z_{in}} \;+\; \frac{1}{z_{iD}}\right). \tag{10}$$

In the following discussion, we assume $z_0 \geq 0$, but it doesn't lose the generality, i.e., one may exchange z for $-z$ in the case of $z_0 \leq 0$.

Let us write down two solutions including the source term separately for two regions, $|z| \leq z_c$ and $|z| \geq z_c$.

$|z| \leq z_c \ (\Lambda_{gc} \leq \Lambda_g)$:

$$\varphi^{(\pm)}_{k,g}(u_g; r_0, u_{g0}) = a^{(\pm)}_g A_{k,g}(\Lambda_g) + b^{(\pm)}_g B_{k,g}(\Lambda_g) - Q_{k,g}(\Lambda_g)\theta(\pm). \tag{11a}$$

$|z| \geq z_c \ (\Lambda_h \leq \Lambda_{hc})$:

$$\varphi^{(\pm)}_{k,h}(u_h; r_0, u_{h0}) = a^{(\pm)}_h A_{k,h}(\Lambda_h) + b^{(\pm)}_h B_{k,h}(\Lambda_h) - Q_{k,h}(\Lambda_h)\theta(\pm). \tag{11b}$$

Here, $Q_{k,i}(\Lambda_i)$ relates to the source term, explicit form of which is presented in Appendix A.

2.2 Boundary condition

We have to find $a^{(\pm)}_g$, $a^{(\pm)}_h$, $b^{(\pm)}_g$ and $b^{(\pm)}_h$ in Eqs. (11a) and (11b), taking into account two continuation conditions at $z = \pm 0$ ($u = r/\bar{r}$) and $z = \pm z_c$ ($u = u_c \equiv r/\bar{r} + |z_c|/\bar{z}$) as well as two boundary conditions at $z = \pm\infty$ ($u = \infty$). Let us summarize the continuation conditions and the boundary conditions explicitly.

$z = \pm 0 \ \ (u = r/\bar{r},\ \Lambda_g = \Lambda_{gr} = \lambda_{g0} U_r)$:

$$\varphi^{(+)}_{k,g}(r/\bar{r}; r_0, u_{g0}) = \varphi^{(-)}_{k,g}(r/\bar{r}; r_0, u_{g0}), \tag{12a}$$

$$\varphi'^{(+)}_{k,g}(r/\bar{r}; r_0, u_{g0}) = \varphi'^{(-)}_{k,g}(r/\bar{r}; r_0, u_{g0}). \tag{12b}$$

$z = \pm z_c \ \ (u = u_c,\ \Lambda_g = \Lambda_{gc},\ \Lambda_h = \Lambda_{hc})$:

$$\varphi^{(\pm)}_{k,g}(u_{gc}; r_0, u_{g0}) = \varphi^{(\pm)}_{k,h}(u_{hc}; r_0, u_{h0}), \tag{13a}$$

$$\varphi'^{(\pm)}_{k,g}(u_{gc}; r_0, u_{g0}) = \varphi'^{(\pm)}_{k,h}(u_{hc}; r_0, u_{h0}). \tag{13b}$$

$z = \pm\infty \ \ (u = \infty,\ \Lambda_h = 0)$:

$$a^{(\pm)}_h A_{k,h}(0) \;+\; b^{(\pm)}_h B_{k,h}(0) \;=\; 0. \tag{14}$$

2.3 Explicit form of the solution

It is very useful to introduce two brackets,

$$(\mathcal{A},\ \mathcal{B})_\pm \;=\; \mathcal{A} \times \mathcal{B}^\dagger \;\pm\; \mathcal{A}^\dagger \times \mathcal{B}\,\bigg|_{z=0}, \tag{15a}$$

$$[\mathcal{A},\ \mathcal{B}] \;=\; \mathcal{A} \times \mathcal{B}^\dagger \;-\; \mathcal{A}^\dagger \times \mathcal{B}\,\bigg|_{z=z_c}. \tag{15b}$$

The meaning of $\dagger$ and the round bracket, $(\cdots)_\pm$, are already appeared in Paper I, which come from the smooth continuation condition of $\varphi^{(+)}_{k,g}$ and $\varphi^{(-)}_{k,g}$ at $z = 0$. Therefore, this bracket is applied only for the term related to the variable $\Lambda_{gr} (= \lambda_{g0} U_g(z = 0))$, and others related to such as Λ_g, Λ_{gc}, Λ_h, Λ_{hc}, ... are freely moved in and out the bracket.

On the other hand, the square bracket, $[\cdots]$, corresponds to the smooth continuation condition of $\varphi^{(\pm)}_{k,g}$ and $\varphi^{(\pm)}_{k,h}$ at $z = \pm z_c$. Therefore, this bracket is applied only for the term related to the variable, Λ_{gc} or Λ_{hc}, and others related to such as Λ_g, Λ_{gr}, Λ_h, ... are freely moved in and out the bracket. In the present works, these brackets play essential role to obtain the solution without complexity. In Appendix B, we demonstrate several examples of these brackets.

These brackets are indeed useful for the mathematical simplification, but the practical procedure of the derivation of the final solution is still cumbersome, and here we give only the result, after integrating over (r_0, z_0), taking account of the source term, $s(r_0, z_0)$, the detail of which will be reported elsewhere. We summarize the solution in the similar form as that presented in the one component model (see Paper I).

$$N(r,u) = \frac{s_{i0}\bar{z}_i^{\,2}}{D_{ii}(r,z)}\sum_{k=1}^{\infty}\mathcal{M}_{k,i}(r)\mathcal{N}_{k,i}(U_i,U_{jc}) + \frac{s_{j0}\bar{z}_j^{\,2}}{D_{ij}(r,z)}\sum_{k=1}^{\infty}\mathcal{M}_{k,j}(r)\mathcal{N}_{k,j}(U_i,U_{jc}), \qquad (16)$$

here

$$D_{ij}(r,z) \;=\; \sqrt{D_i(0,z)D_j(\kappa_j r,0)}, \qquad (17)$$

$$D_i(r,z) = D_{i0}\exp[r/r_D + |z|/z_{iD}], \qquad (18)$$

$$\kappa_i = \left(1+\frac{r_D}{r_n}\right)\Big/\left(1+\frac{z_{iD}}{z_{in}}\right) \simeq 1, \qquad (19)$$

$$[i,\ j] \;\equiv\; \left\{\begin{array}{ll} [\text{“}g\text{”},\ \text{“}h\text{”}], & \text{for } |z|\le z_c \\ [\text{“}h\text{”},\ \text{“}g\text{”}], & \text{for } |z|\ge z_c \end{array}\right\} \qquad (20)$$

Here, we should remember that bounded to Eqs. (4a) $\sim$ (4c) is each pair for the three critical parameters, $[D_{g0},\, D_{h0}]$, $[n_{g0},\, n_{h0}]$, and $[s_{g0},\, s_{h0}]$, respectively.

Corresponding to Eqs. (29), (30) and (31) in Paper I, we have

$$\mathcal{M}_{k,i}(r) \simeq \mathcal{M}_k(r) \simeq \frac{2J_0(\xi_k r/R)}{\xi_k J_1(\xi_k)}, \qquad (21)$$

$$\mathcal{N}_{k,i}(X,Y) = \int_0^1 t^{\omega_i-1}\,\Psi_{k,i}(t,X,Y)\,\Theta_i(t)dt, \qquad (22)$$

$$\text{with} \quad \omega_i \;=\; 2\omega_{\perp i} - \nu_i \;\simeq\; 2-\nu_i, \qquad (23)$$

where

$$\Theta_i(t) = \left\{\begin{array}{ll} \theta(t-U_{gc}), & \text{for } i\equiv \text{“}g\text{”} \\ \theta(U_{hc}-t), & \text{for } i\equiv \text{“}h\text{”} \end{array}\right\} \qquad (24)$$

$\Psi_{k,i}$ in Eq. (22) is summarized in Appendix C.

One may note Eq. (16) is quite similar to Eq. (28$'$) in Paper I. In fact, we find that they coincide completely with each other in both limits of $z_c \to \infty$ and $z_c \to 0$ (see Appendix D).

2.4 Solution at the galactic plane

Let us present explicitly the solution in the case of the galactic plane ($z=0$ or equivalently $u=u_r=r/\bar{r}$). In the case of $z=0$, we have

$$\Psi_{k,g}(t,U_r,U_{hc}) = \frac{[\mathcal{L}_{k,g}(\Lambda_{gc},\Lambda_{gt}), I_{\nu_{k,h}}(\Lambda_{hc})]}{[I_{\nu_{k,h}}(\Lambda_{hc}), \mathcal{L}^{\dagger}_{k,g}(\Lambda_{gr},\Lambda_{gc})]}, \qquad (25a)$$

$$\Psi_{k,h}(t,U_r,U_{hc}) = \frac{I_{\nu_{k,h}}(\Lambda_{ht})}{[I_{\nu_{k,h}}(\Lambda_{hc}), \mathcal{L}^{\dagger}_{k,g}(\Lambda_{gr},\Lambda_{gc})]}, \qquad (25b)$$

$$\text{with} \quad \Lambda_{gt} = \lambda_{g0}t, \quad \Lambda_{ht} = \lambda_{h0}t.$$

Taking care of the constraint condition Eq. (4), finally we obtain a quite similar form of the solution as that in the case of one component model (Eq. (34) in Paper I),

$$N(r,r/\bar{r}) \simeq \frac{s_{g0}\bar{z}_g^{\,2}}{D_g(r/2,0)}\sum_{k=1}^{\infty}\frac{2J_0(\xi_k r/R)}{\xi_k J_1(\xi_k)}\frac{\mathcal{H}_k(\lambda_{g0},\lambda_{h0})}{H_k^{\dagger}(\Lambda_{gr})}, \qquad (26)$$

where the explicit forms of $\mathcal{H}_k(a,b)$ and $H_k^{\dagger}(\Lambda)$ are presented in Appendix D. For the limit of $z_c \to \infty$ or $z_c \to 0$, Eq. (26) is completely coincident with that of Eq. (34) of paper I, as shown in Appendix D.

3 Discussion

We obtained the solution of 3D cosmic-ray diffusion with two component scale heights, one corresponding to the disk and the another to the halo. The solution is of quite similar form as that in the case of one component scale height. We confirmed the present solution is completely coincident with that of the one component model for $z_c \to \infty$ and/or $z_c \to 0$.

We found that two kinds of brackets, $(\cdots)_{\pm}$ and $[\cdots]$, are quite useful in order to obtain the solution of diffusion equation without the complexity in the procedure of the evaluation.

In the present report, we focused to the procedure of the derivation of the solution, and numerical results will be reported elsewhere. For the practical application of the present result, we have to obtain further various observables, such as the path length distribution, secondary/primary, isotope, diffused γ, and so on, but we don't touch them due to the limited space, some of which might be reported at the oral session in the conference.

Appendix A Explicit form of $Q_{k,i}(\Lambda_i)$

This function is related to the source term $q_{k,i}(r_0,\ z_0)$ (see Eq. (13) in Paper I).

$$Q_{k,i}(\Lambda_i) = q_{k,i}(r_0,z_0)\Lambda_{i0}^{-2\nu_i}L_{k,i}(\Lambda_i,\Lambda_{i0})\Theta(\Lambda_i,\Lambda_{i0},\Lambda_{ic}), \qquad (A1)$$

$$\text{with} \quad \nu_i = \frac{\bar{z}_i}{2z_{iD}} = 1\Big/\left(1+\frac{z_{iD}}{z_{in}}\right), \qquad (A2)$$

$$q_{k,i}(r_0,z_0) = \frac{\bar{z}_i}{D_i(r_0,z_0)}\frac{J_0(\xi_k r_0/R)}{\pi R^2 J_1^2(\xi_k)}. \qquad (A3)$$

$$L_{k,i}(X,Y) = A_{k,i}(X)B_{k,i}(Y) - A_{k,i}(Y)B_{k,i}(X), \qquad (A4)$$

$$\Theta(\Lambda_i,\Lambda_{i0},\Lambda_{ic}) \;=\; \left\{\begin{array}{ll} \theta(\Lambda_g-\Lambda_{g0})\theta(\Lambda_{g0}-\Lambda_{gc}), & \text{for} \quad i \;\equiv\; \text{“}g\text{”} \\ \theta(\Lambda_h-\Lambda_{h0})\theta(\Lambda_{hc}-\Lambda_h), & \text{for} \quad i \;\equiv\; \text{“}h\text{”} \end{array}\right\} \qquad (A5)$$

Appendix B Examples of the sqaure bracket

Here we give two examples appeared in the numerator and the denominator of the righhand side of Eq. (25a).

$$[\mathcal{L}_{k,g}(\Lambda_{gc},\Lambda_{gt}),\ I_{\nu_{k,h}}(\Lambda_{hc})] \ = $$
$$\mathcal{L}_{k,g}(\Lambda_{gc},\Lambda_{gt})I^{\dagger}_{\nu_{k,h}}(\Lambda_{hc}) - \mathcal{L}^{\dagger}_{k,g}(\Lambda_{gc},\Lambda_{gt})I_{\nu_{k,h}}(\Lambda_{hc})$$
$$= \ H^{(2)}_{k,1}I_{\nu_{k,g}}(\Lambda_{gt}) - H^{(1)}_{k,1}K_{\nu_{k,g}}(\Lambda_{gt}), \qquad (B1a)$$

$$[I_{\nu_{k,h}}(\Lambda_{hc}),\ \mathcal{L}^{\dagger}_{k,g}(\Lambda_{gr},\Lambda_{gc})] \ = $$
$$I_{\nu_{k,h}}(\Lambda_{hc})\mathcal{L}^{\dagger\dagger}_{k,g}(\Lambda_{gr},\Lambda_{gc}) - I^{\dagger}_{\nu_{k,h}}(\Lambda_{hc})\mathcal{L}^{\dagger}_{k,g}(\Lambda_{gr},\Lambda_{gc})$$
$$= \ H^{(2)}_{k,1}I^{\dagger}_{\nu_{k,g}}(\Lambda_{gr}) - H^{(1)}_{k,1}K^{\dagger}_{\nu_{k,g}}(\Lambda_{gr}). \qquad (B1b)$$

$$\mathcal{L}^{\dagger\dagger}_{k,g}(X,Y) = I^{\dagger}_{\nu_{k,g}}(X)K^{\dagger}_{\nu_{k,g}}(Y) - I^{\dagger}_{\nu_{k,g}}(Y)K^{\dagger}_{\nu_{k,g}}(X). \qquad (B2)$$

$I^{\dagger}_{\nu}(\Lambda)$, $K^{\dagger}_{\nu}(\Lambda)$, and $\mathcal{L}_{k,g}(X,Y)$, $\mathcal{L}^{\dagger}_{k,g}(X,Y)$ are presented in Eqs. (A1a), (A1b), and Eqs. (A4a), (A4b) in Paper I, and $H^{(1)}_{k,1}, H^{(2)}_{k,1}$ are defined in Appendix D in this paper.

It should be remarked that

$$\mathcal{L}_{k,g}(\Lambda,\Lambda) = 0, \ \ \mathcal{L}^{\dagger}_{k,g}(\Lambda,\Lambda) = 1, \ \ \mathcal{L}^{\dagger\dagger}_{k,g}(\Lambda,\Lambda) = 0. \qquad (B3)$$

Appendix C Definition of the exchange-function

$$\psi_{k,i}(X,Y,Z) \ = \ \frac{[\mathcal{L}_{k,i}(\Lambda_{ic},\lambda_{i0}X),\ I_{\nu_{k,h}}(\lambda_{h0}Y)\mathcal{L}^{\dagger}_{k,g}(\Lambda_{gr},\lambda_{g0}Z)]}{[I_{\nu_{k,h}}(\Lambda_{hc}),\ \mathcal{L}^{\dagger}_{k,g}(\Lambda_{gr},\Lambda_{gc})]}, \quad (C1)$$

<u>for $i \neq j$;</u>

$$\Psi_{k,i}(X,Y,U_{jc}) \ = \ \left\{ \begin{array}{l} \left\{ \begin{array}{ll} \psi_{k,g}(X,U_{hc},Y), & \text{for } X \le Y \\ \psi_{k,g}(Y,U_{hc},X), & \text{for } X \ge Y \end{array} \right\} : i,j \equiv \text{“}g\text{”, “}h\text{”} \\ \\ \left\{ \begin{array}{ll} \psi_{k,h}(Y,X,U_{gc}), & \text{for } X \le Y \\ \psi_{k,h}(X,Y,U_{gc}), & \text{for } X \ge Y \end{array} \right\} : i,j \equiv \text{“}h\text{”, “}g\text{”} \end{array} \right\} \qquad (C2)$$

<u>for $i = j$;</u>

$$\Psi_{k,i}(X,Y,U_{ic}) = \left\{ \begin{array}{ll} \psi_{k,g}(U_{gc},Y,X) & : i \equiv \text{“}g\text{”} \\ \psi_{k,h}(U_{hc},X,Y) & : i \equiv \text{“}h\text{”} \end{array} \right\} \quad (C3)$$

Appendix D Explicit forms of $H^{\dagger}_{k}$ and $\mathcal{H}_k$

We introduce following variables and functions,

$$H^{(1)}_{k,1} = \ [I_{\nu_{k,h}}(\Lambda_{hc}),\ I_{\nu_{k,g}}(\Lambda_{gc})], \qquad (D1a)$$
$$H^{(2)}_{k,1} = \ [I_{\nu_{k,h}}(\Lambda_{hc}),\ K_{\nu_{k,g}}(\Lambda_{gc})], \qquad (D1b)$$
$$H^{(1)}_{k,2} = \ [K_{\nu_{k,h}}(\Lambda_{hc}),\ I_{\nu_{k,g}}(\Lambda_{gc})], \qquad (D1c)$$
$$H^{(2)}_{k,2} = [K_{\nu_{k,h}}(\Lambda_{hc}),\ K_{\nu_{k,g}}(\Lambda_{gc})]. \qquad (D1d)$$

$$H^{\dagger}_{k}(\Lambda) \ = \ H^{(2)}_{k,1}I^{\dagger}_{\nu_{k,g}}(\Lambda) - H^{(1)}_{k,1}K^{\dagger}_{\nu_{k,g}}(\Lambda), \qquad (D2)$$

$$\mathcal{H}_k(a,b) = H^{(2)}_{k,1}\mathcal{I}_{\nu_{k,g}}(a) - H^{(1)}_{k,1}\mathcal{K}_{\nu_{k,g}}(a) + H_0\mathcal{I}_{\nu_{k,h}}(b), \qquad (D3)$$

where

$$\mathcal{I}_{\nu_{k,g}}(a) \ = \ \int_{U_{gc}}^{1} t^{\omega_g - 1}I_{\nu_{k,g}}(at)dt, \qquad (D4a)$$
$$\mathcal{K}_{\nu_{k,g}}(a) = \int_{U_{gc}}^{1} t^{\omega_g - 1}K_{\nu_{k,g}}(at)dt, \qquad (D4b)$$
$$\mathcal{I}_{\nu_{k,h}}(b) = \int_{0}^{U_{hc}} t^{\omega_h - 1}I_{\nu_{k,h}}(bt)dt, \qquad (D4c)$$

and with use of the constraint conditions Eqs. (4a) and (4c),

$$H_0 = \frac{\bar{z}_h^2}{\bar{z}_g^2}\frac{s_{h0}}{s_{g0}}\sqrt{\frac{D_{g0}}{D_{h0}}} = \frac{\bar{z}_h^2}{\bar{z}_g^2}\mathrm{e}^{-z_c/\hat{z}}, \qquad (D5)$$

$$\text{with} \ \ \frac{1}{\hat{z}} = \left(\frac{1}{z_{gs}} + \frac{1}{2z_{gD}}\right) - \left(\frac{1}{z_{hs}} + \frac{1}{2z_{hD}}\right). \qquad (D6)$$

Let us check the consistency of the present solution with that in the case of one component model for $z_c \to \infty$ and/or $z_c \to 0$. For instance, in the case of $z_c \to \infty$, we find immediately

$$U_{gc}, U_{hc} \to 0, \quad \Lambda_{gc}, \Lambda_{hc} \to 0,$$

leading to

$$H^{(1)}_{k,1} \to 0, \quad H^{(2)}_{k,1} \to \infty,$$

i.e.;

$$\mathcal{H}_k(\lambda_{g0},\lambda_{h0}) \to H^{(2)}_{k,1}\mathcal{I}_{\nu_{k,g}}(\lambda_{g0}),$$
$$H^{\dagger}_{k}(\Lambda_{gr}) \to H^{(2)}_{k,1}I^{\dagger}_{\nu_{k,g}}(\Lambda_{gr}).$$

Then we obtain

$$\frac{\mathcal{H}_k(\lambda_{g0},\lambda_{h0})}{H^{\dagger}_{k}(\Lambda_{gr})} \ \Rightarrow \ \frac{\mathcal{I}_{\nu_k}(\lambda_{g0})}{I^{\dagger}_{\nu_k}(\Lambda_{gr})},$$

which coincides with the solution of one component model.

In the case of $z_c \to 0$, we find also it gives the same solution as that of the one component model.

References

T. Shibata, Analytical solution of 3D cosmic-ray diffusion in boundaryless halo (I) — one component model —, in this volume.