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Analytical solution of 3D cosmic-ray diffusion in boundaryless halo (III) – secondary components –

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Abstract. Based on the solution of three dimensional cosmic-ray diffusion for the primary component derived by the author, taking a rather realistic structure of our Galaxy into account, we show an analytical solution of the cosmic-ray diffusion for secondary components, typically for Li-Be-B mainly coming from C and O, and for sub-Fe coming from Fe. We present also the solution for unstable nuclei, typically ^{10}Be.

1 Introduction

In two papers (Shibata I, II, 2001; hereafter called Paper I and Paper II) presented in this volume, we showed the solution of 3D cosmic-ray (CR) diffusion, taking a rather realistic structure of our Galaxy into account, where we assumed three critical parameters, the diffusion coefficient D, the gas density n and the CR source density s, depend on the position (r, z) in cylindrical cordinate, each having exponential distribution function, while in Paper II, each one has two component scale-heights, one corresponding to the disk and the other to the halo.

In the present paper, we give the solution for secondary components with the use of the one component model, both stable and the unstable nuclei, coming from the fragmentation of the primary components. It is needless to say that these data bring us invaluable informations in understanding the CR origin and the mechanism of the acceleration and the propagation in our Galaxy.

Though many elaborate works have been reported in the past (Berezinskii et al. 1990), there still remains unclear problems, far from the unified picture for the propagation of the all kinds of CR components, both 1-ry and 2-ry. For instance, according to several recent works (Silberberg et al. 1993; Seo et al. 1994; Heinbach et al. 1995; Mitsui 1996; Nishimura 1997), the reacceleration process is not always effective, and a simple leaky box model is rather in good agreement with the data, while it is quite likely that the scattering of CR due to the random magnetic field must happen during the propagation of CR in the Galaxy.

In order to study these difficulties, we propose a new solution for the CR diffusion problem, taking various effects, spatial dependency of the diffusion coeficient, the gas density and so on.

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2 Solution for stable nucleus

In Paper I, we showed the explicit form of the solution of CR diffusion, $\varPhi_P^{(\pm)}(r, u; r_0, u_0)$, where $u = r/\bar{r} + z/\bar{z}$, $u_0 = r_0/\bar{r} + z_0/\bar{z}$, and also the CR intensity of the primary component $N_P(r, u)$, which is obtained by the integration with the weight of CR source density $s(r_0, z_0)$,

$$N_P(r, u) = \int_0^R 2\pi r_0 dr_0 \int_{-\infty}^{+\infty} dz_0 s(r_0, z_0)\, \varPhi_P^{(\pm)}(r, u; r_0, u_0). \quad (1)$$

Notations used in the present paper are the same as those in Paper I, though sometimes attaching here the subscripts, "P" and "S", in order to discriminate the primary and the secondary components.

Secondary components originated from the above primary component $N_P(r, u)$ are then straightforwardly given by

$$N_S(r, u) = \int_0^R 2\pi r_0 dr_0 \int_{-\infty}^{+\infty} dz_0 \times [N_P(r_0, u_0) n(r_0, z_0) v \sigma_{PS}]\, \varPhi_S^{(\pm)}(r, u; r_0, u_0). \quad (2)$$

Here, σ_{PS} denotes the fragmentaion cross section, $P \rightarrow S$, and we omit the contribution from the second step product (second generation fragments), $P \rightarrow S' \rightarrow S$. Contribution of the second and higher generation fragments will be discussed elsewhere.

Explit forms of N_P and $\Phi_S^{(\pm)}$ (see Paper I) are

$$N_P(r_0,u_0)=\frac{s_0\bar{z}^2}{D_0}U_0^{\nu}\sum_{k=1}^{\infty}\mathcal{M}_k(r_0)\mathcal{N}_k(U_0,U_{r_0}), \quad (3)$$

$$\Phi_S^{(\pm)}(r,u;r_0,u_0)=\left(\frac{U}{U_0}\right)^{\nu}\sum_{m=1}^{\infty}q_m(r_0,z_0)J_0(\zeta_m r/R)\times$$
$$\left[\frac{(I_{\nu_m}(\Lambda_r^*)I_{\nu_m}(\Lambda^*),\mathcal{L}_m(\Lambda_r^*,\Lambda_0^*))_{\pm}}{(I_{\nu_m}(\Lambda_r^*),I_{\nu_m}(\Lambda_r^*))_{+}}-\mathcal{L}_m(\Lambda^*,\Lambda_0^*)\theta(u_0-u)\right] \quad (4)$$

where ζ_m $(m=1,2,\ldots)$ are the roots of $J_0(\zeta_m)=0$,

$$\nu_m^2 = \nu^2+\lambda_{1,m}^2, \quad (5)$$

$$\text{with}\ \ \nu=\frac{\bar{z}}{2z_D},\quad \lambda_{1,m}=\frac{\zeta_m\bar{z}}{R}, \quad (6)$$

and

$$\Lambda^*=\lambda_0^*U,\quad \Lambda_0^*=\lambda_0^*U_0,\quad \Lambda_r^*=\lambda_0^*U_r, \quad (7)$$

$$\text{with}\quad \lambda_0^*=\sqrt{\frac{n_0 v\sigma_S}{D_0}}\,\bar{z}. \quad (8)$$

σ_S is the cross section for the secondary concerned, and all the variables, U, U_0, U_r, $q_m(r_0,z_0)$, ... are the same as those defined in Paper I.

Performing the integrations with respect to r_0 and z_0, we obtain a similar form as eq. (3)

$$N_S(r,u)=f_{PS}\lambda_0^2\frac{s_0\bar{z}^2}{D_0}U^{\nu}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\mathcal{M}_{k,m}(r)\mathcal{N}_{k,m}(U,U_r), \quad (9)$$

where explicit forms of $\mathcal{M}_{k,m}$ and $\mathcal{N}_{k,m}$ are summarized in Appendix A, and

$$\lambda_0=\sqrt{\frac{n_0 v\sigma_P}{D_0}}\,\bar{z}\quad\text{and}\quad f_{PS}=\frac{\sigma_{PS}}{\sigma_P}. \quad (10)$$

f_{PS} corresponds to the production rate of the fragment S in the collision of the primary P with ISM.

Finally, we obtain the relative intensity of the secondary to the primary,

$$\frac{N_S}{N_P} = f_{PS}\lambda_0^2\frac{\sum_k\sum_m\mathcal{M}_{k,m}(r)\mathcal{N}_{k,m}(U,U_r)}{\sum_k\mathcal{M}_k(r)\mathcal{N}_k(U,U_r)}. \quad (11)$$

3 Solution for unstable nucleus

In this section, we present the solution in the case of unstable nucleus, though we don't touch the effect of the energy loss as well as that of the reacceleration during the propagation in the Galaxy, because of the limited space.

The procedure of the derivation of the solution is the same as that presented in the last section, but $\Phi_S^{(\pm)}$ must be changed slightly. If we don't take the energy change during the propagation into account, it is straightforward to write down the diffusion equation, i.e., Eq. (10) in Paper I is replaced by

$$\left[\frac{d^2}{du^2}+2\nu\frac{d}{du}-(\lambda_0^{*2}\mathrm{e}^{-2u}+\lambda_\tau^2\mathrm{e}^{-2\nu u}+\lambda_{1,m}^2)\right]\varphi_m$$
$$=-q_m(r_0,z_0)\delta(u-u_0), \quad (12)$$

$$\text{with}\quad \lambda_\tau=\frac{\bar{z}}{\sqrt{\tau D_0}}, \quad (13)$$

where $\tau=\gamma\tau_0$ (τ_0: rest lifetime, γ: Lorentz factor).

Since it is somewhat complicated to show general analytical solution in this case, we present three typical cases in choice of two scale heights, z_D and z_n, which are both critical parameters in CR propagation,

$$\text{(a)}\ z_D\gg z_n\ \Rightarrow\ \nu=0,\ \ \bar{z}=2z_n\ll z_D,$$

$$\text{(b)}\ z_D\simeq z_n\ \Rightarrow\ \nu=\tfrac{1}{2},\ \ \bar{z}=z_n=z_D,$$

$$\text{(c)}\ z_D\ll z_n\ \Rightarrow\ \nu=1,\ \ \bar{z}=2z_D\ll z_n.$$

3.1 Solution in the case of $\nu=0$ ($\bar{z}=2z_n$)

In this case, we obtain completely the same solution as Eq. (9), if we change ν_m defined by Eq. (5) with

$$\nu_m^2=\lambda_\tau^2+\lambda_{1,m}^2. \quad (14)$$

3.2 Solution in the case of $\nu=1$ ($\bar{z}=2z_D$)

In this case also, we obtain the same solution as Eq. (9), if we change λ_0^* defined by Eq. (8) with

$$\lambda_0^*=\lambda_\tau\sqrt{n_0 v\sigma_S\tau+1}. \quad (15)$$

3.3 Solution in the case of $\nu=1/2$ ($\bar{z}=z_D=z_n$)

In this case, the situation is somewhat different from the above two cases, and with use of the variable $U(=\mathrm{e}^{-u})$ in place of u, Eq. (12) is now reduced to

$$\left[\frac{d^2}{dU^2}-\left(\lambda_0^{*2}+\frac{\lambda_\tau^2}{U}+\frac{\lambda_{1,m}^2}{U^2}\right)\right]\varphi_m=-q_m\delta(U-U_0). \quad (16)$$

Fundamental solution of this equation is given by use of the Confluent Hypergeometrical function $F(a,b;x)$, and putting simply

$$F_{\mu,\nu}(x)=F(\mu+\nu,1+2\nu;2x), \quad (17)$$

we have, (see Eq. (14) in Paper I)

$$\varphi_m^{(\pm)}(u;r_0,u_0)=a^{(\pm)}A_m(\Lambda^*)+b^{(\pm)}B_m(\Lambda^*), \quad (18)$$

where Λ^* is defined by Eqs. (7) and (8), and

$$A_m(\Lambda^*)=2\sqrt{\nu_m\Lambda^*}\,F_{\nu_m}(\Lambda^*), \quad (19a)$$

$$B_m(\Lambda^*)=2\sqrt{\nu_m\Lambda^*}\,G_{\nu_m}(\Lambda^*), \quad (19b)$$

and

$$F_\nu(x) = \frac{1}{\sqrt{2\nu}} \mathrm{e}^{-x}(2x)^{+\nu} F_{\mu,+\nu}(x), \qquad (20a)$$

$$G_\nu(x) = \frac{1}{\sqrt{2\nu}} \mathrm{e}^{-x}(2x)^{-\nu} F_{\mu,-\nu}(x), \qquad (20b)$$

$$\mu = \mu_\tau + 1/2 \quad \text{with} \quad \mu_\tau = \lambda_\tau^2/2\lambda_0^*. \qquad (21)$$

One may remark that $F_{\nu_m}(\Lambda^*)$ and $G_{\nu_m}(\Lambda^*)$ introduced above correspond to $I_{\nu_m}(\Lambda)$ and $K_{\nu_m}(\Lambda)$ respectively, appeared in the solution in the case of the stable nucleus. In fact, they resemble greatly each other as shown in Appendix B.

Now, taking acccount of the boundary conditions, we obtain

$$\Phi_S^{(\pm)}(r,u;r_0,u_0) = \left(\frac{U}{U_0}\right)^{\frac{1}{2}} \sum_{m=1}^{\infty} q_m(r_0,z_0) J_0(\zeta_m r/R) \times$$

$$\left[\frac{(F_{\nu_m}(\Lambda_r^*)F_{\nu_m}(\Lambda^*), \mathcal{L}_m(\Lambda_r^*, \Lambda_0^*))_\pm}{(F_{\nu_m}(\Lambda_r^*), F_{\nu_m}(\Lambda_r^*))_+} - \mathcal{L}_m(\Lambda^*, \Lambda_0^*)\theta(u_0 - u)\right] \qquad (22)$$

$$\mathcal{L}_m(X,Y) = F_{\nu_m}(X)G_{\nu_m}(Y) - F_{\nu_m}(Y)G_{\nu_m}(X), \quad (23a)$$

$$\mathcal{L}_m^\dagger(X,Y) = F_{\nu_m}^\dagger(X)G_{\nu_m}(Y) - F_{\nu_m}(Y)G_{\nu_m}^\dagger(X). \quad (23b)$$

Eq. (22) is completely the same form as Eq. (4) after replacing I_{ν_m}, K_{ν_m} and ν by F_{ν_m}, G_{ν_m} and 1/2 respectively. In Appendix B, we present the explicit forms of the brackets appeared in Eq. (22) and compare them with those found in the case of the stable nucleus.

Now integrating over r_0 and z_0 with the weight of the source, $[nv\sigma_{PS}]N_P$, (see Eq. (2)), we obtain

$$N_S^{(\tau)}(r,u) = f_{PS}^{(\tau)} \lambda_0^2 \frac{s_0 \bar{z}^2}{D_0} U^{\frac{1}{2}} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \mathcal{M}_{k,m}(r) \mathcal{N}_{k,m}^{(\tau)}(U,U_r), \qquad (24)$$

$$\text{with} \quad f_{PS}^{(\tau)} = \sigma_{PS}^{(\tau)}/\sigma_P, \qquad (25)$$

that is, $f_{PS}^{(\tau)}$ is the production rate of the unstable nucleus. Explicit form of $\mathcal{N}_{k,m}^{(\tau)}$ is summarized in Appendix C, while $\mathcal{M}_{k,m}(r)$ is the same as that used in the case of stable nucleus.

The relative intensity of the unstable nucleus to the stable nucleus in the case of $\nu = 1/2$ is given by

$$\frac{N_S^{(\tau)}}{N_S} = f_\tau \frac{\sum_k \sum_m \mathcal{M}_{k,m}(r)\mathcal{N}_{k,m}^{(\tau)}(U,U_r)}{\sum_k \sum_m \mathcal{M}_{k,m}(r)\mathcal{N}_{k,m}(U,U_r)}, \qquad (26)$$

$$\text{with} \quad f_\tau = \sigma_{PS}^{(\tau)}/\sigma_{PS}^{(\tau=\infty)}, \qquad (27)$$

where f_τ is the relative production rate of the unstable nucleus to the stable one.

4 Summary

In the present and the other two papers (Shibata I, II 2001) appeared in this volume, we obtained analytical solutions for 3D cosmic-ray diffusion in our Galaxy, taking account of rather realistic effects such as the spatial dependence for three critical parameters, diffusion coefficient D, the gas density n and the source density s.

We found the solution for the diffusion of unstable nucleus is obtained by replacing the variables and functions used in the case of the stable nucleus with following ones for three typical ν's;

(a) $\nu = 0$ $(\bar{z} = 2z_n \ll z_D)$: $\nu_m^2 \Rightarrow \lambda_\tau^2 + \lambda_{1,m}^2$,

(b) $\nu = \frac{1}{2}$ $(\bar{z} = z_n = z_D)$: $I_{\nu_m}, K_{\nu_m} \Rightarrow F_{\nu_m}, G_{\nu_m}$,

(c) $\nu = 1$ $(\bar{z} = 2z_D \ll z_n)$: $\lambda_0^* \Rightarrow \lambda_\tau \sqrt{n_0 v \sigma_S \tau + 1}$.

Unfortunately, we are not in time for the numerical results, and for the comparison with recent experimental data on various kinds of components (see references appeared in Paper I), and couldn't touch the low energy effect, particularly ionization loss, and the reacceleration process. These low energy effects are not so simple as discussed here, since the diffusion equation is in general not separable in space and energy.

About this problem, Ptsukin et al. pointed out recently a possibility to use the weighted slab approximation (Ptsukin et al. 1996) under some condition. Based on their consideration, we will discuss these problems elsewhere.

Appendix A Explcit forms of $\mathcal{M}_{k,m}$ and $\mathcal{N}_{k,m}$

$$\mathcal{M}_{k,m}(r) = \frac{\Xi(\xi_k, R/\hat{r})}{J_1^2(\xi_k)} \frac{J_0(\zeta_m r/R)}{J_1^2(\zeta_m)}, \qquad (A1)$$

$$\mathcal{N}_{k,m}(X,Y) = \int_0^1 \Psi_m(y,X,Y) y dy \int_0^1 x^{\omega-1} \bar{\Psi}_{k,m}(x,y) dx. \qquad (A2)$$

Here $\omega \simeq 2 - \nu$ (see Paper I), and Ψ_m is the exchange-function defined by Eqs. (26) and (27) in Paper I after replacing k, λ_0 by m, λ_0^*, respectively,

$$\Psi_m(X,Y,Z) = \left\{ \begin{array}{ll} \psi_m(X,Y,Z), & \text{for } X \le Y \\ \psi_m(Y,X,Z), & \text{for } X \ge Y \end{array} \right\} \quad (A3)$$

$$\psi_m(X,Y,Z) = \mathcal{L}_m^\dagger(\lambda_0^* Z, \lambda_0^* Y) \frac{I_{\nu_m}(\lambda_0^* X)}{I_{\nu_m}^\dagger(\lambda_0^* Z)}, \qquad (A4)$$

where $\mathcal{L}_m^\dagger(X,Y)$ is obtained by replacing F_{ν_m}, G_{ν_m} in Eq. (23b) with I_{ν_m}, K_{ν_m} respectively. $\bar{\Psi}_{k,m}(X,Y)$ is also the exchange-function defined by,

$$\bar{\Psi}_{k,m}(X,Y) = \left\{ \begin{array}{ll} \bar{\psi}_{k,m}(X,Y), & \text{for } X \le Y \\ \bar{\psi}_{k,m}(Y,X), & \text{for } X \ge Y \end{array} \right\} \quad (A5)$$

where

$$\bar{\psi}_{k,m}(X,Y) = \int_0^1 \psi_k(X,Y,U_t) J_0(\xi_k t) J_0(\zeta_m t) t dt, \quad (A6)$$

$$\text{with} \quad U_t = \mathrm{e}^{-[R/\bar{r}]t}, \quad (A7)$$

where $\psi_k(X,Y,Z)$ is defined by Eq. (27) in Paper I, i.e., the same form as the above Eq. ($A4$) after replacing m, λ_0^* by k, λ_0, respectively. Then Eq. ($A6$) is rewritten,

$$\bar{\psi}_{k,m}(X,Y) = I_{\nu_k}(\lambda_0 X)[c_{k,m} K_{\nu_k}(\lambda_0 Y) - d_{k,m} I_{\nu_k}(\lambda_0 Y)], \quad (A6')$$

where

$$c_{k,m} = \int_0^1 J_0(\xi_k t) J_0(\zeta_m t) t dt, \quad (A8a)$$

$$d_{k,m} = \int_0^1 J_0(\xi_k t) J_0(\zeta_m t) \frac{K^\dagger_{\nu_k}(\lambda_0 U_t)}{I^\dagger_{\nu_k}(\lambda_0 U_t)} t dt. \quad (A8b)$$

Appendix B Summary of $F_\nu(x)$ and $G_\nu(x)$

Here we summarize two functions, $F_\nu(x)$ and $G_\nu(x)$, expressed with use of the Confluent Hypergeometrical function, $F(a,b;\ x)$.

Let us introduce a function,

$$F^\dagger_{\mu,\nu}(x) = (\mu+\nu) F_{\mu+1,\nu}(x) - (\mu_\tau + x) F_{\mu,\nu}(x), \quad (B1)$$

which corresponds to $I^\dagger_\nu(x)$ and/or $K^\dagger_\nu(x)$ appeared often in the solution for the stable nucleus, and put

$$F^\dagger_\nu(x) = \frac{1}{\sqrt{2\nu}} \mathrm{e}^{-x} (2x)^{+\nu} F^\dagger_{\mu,+\nu}(x), \quad (B2a)$$

$$G^\dagger_\nu(x) = \frac{1}{\sqrt{2\nu}} \mathrm{e}^{-x} (2x)^{-\nu} F^\dagger_{\mu,-\nu}(x). \quad (B2b)$$

With use of these functions, we find

$$\frac{d}{du} A_m(\Lambda^*) = -2\sqrt{\nu_m \Lambda^*}\, F^\dagger_{\nu_m}(\Lambda^*), \quad (B3a)$$

$$\frac{d}{du} B_m(\Lambda^*) = -2\sqrt{\nu_m \Lambda^*}\, G^\dagger_{\nu_m}(\Lambda^*). \quad (B3b)$$

One should note that the Wronskian, $W(F_\nu, G_\nu)$, is given by

$$W(F_\nu, G_\nu) = F_\nu(x) G'_\nu(x) - F'_\nu(x) G_\nu(x) = -\frac{1}{x}, \quad (B4a)$$

corresponding to

$$W(I_\nu, K_\nu) = I_\nu(x) K'_\nu(x) - I'_\nu(x) K_\nu(x) = -\frac{1}{x}. \quad (B4b)$$

So, in the case of $X = Y = \mathit{\Lambda}_r$ in Eq. ($23b$), we find immediately

$$\mathcal{L}^\dagger_m(\mathit{\Lambda}_r, \mathit{\Lambda}_r) = 1, \quad (B5)$$

i.e., the above relation holds both for the Modified Bessel function and the Confluent Hypergeometrical function. These relations are quite useful for the practical calculation at the galactic plane, $z = 0$.

Appendix C Explicit form of $\mathcal{N}^{(\tau)}_{k,m}$

As discussed in Appendix B, the difference of the solution between the stable and the unstable nuclei is found in ψ_m only, the explicit form of which is presented in Eq. ($A4$) in the case of the stable nucleus. One should remark it is also the same form as in the case of the primary component (see Eq. (27) in Paper I) after replacing λ_0 by λ_0^*, or equivalently replacing σ_P by σ_S.

Then the form of $\mathcal{N}^{(\tau)}_{k,m}(X,Y)$ is completely the same as Eq. ($A2$) after redefining the exchange-function

$$\psi_m(X,Y,Z) = \mathcal{L}^\dagger_m(\lambda_0^* Z, \lambda_0^* Y) \frac{F_{\nu_m}(\lambda_0^* X)}{F^\dagger_{\nu_m}(\lambda_0^* Z)}, \quad (C1)$$

where $\mathcal{L}^\dagger_m(X,Y)$ is defined by Eq. ($23b$).

In the case of the galactic plane ($Y = Z = U_r$), we have a simple result with use of Eq. ($B5$),

$$\psi_m(X, U_r, U_r) = \frac{F_{\nu_m}(\lambda_0^* X)}{F^\dagger_{\nu_m}(\lambda_0^* U_r)}, \quad (C2)$$

while for the stable nucleus or the primary component,

$$\psi_m(X, U_r, U_r) = \frac{I_{\nu_m}(\lambda_0^* X)}{I^\dagger_{\nu_m}(\lambda_0^* U_r)}. \quad (C3)$$

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