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Stationary cosmic ray transport in the Galaxy

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Abstract. This is a progress report on Galactic cosmic ray propagation calculations. To model the spectra of primary and secondary cosmic rays in the solar vicinity we use an analytical solution of the stationary transport equation for Galactic cosmic rays in cylindrical geometry. In particular stochastic reacceleration of the particles by scattering through plasma waves in the interstellar medium is considered. Consistency checks of the solution and first studies regarding the cosmic ray protons are presented.

1 Introduction

To explain the measured hadronic cosmic ray energy spectra above 1 GeV at solar position (e.g. Engelmann et al. (1990), Menn et al. (2000), Sanuki et al. (2000), Wiebel-Sooth et al. (1998), Webber (1997)) the propagation, acceleration and interaction of Galactic cosmic rays in the interstellar medium and in the Galactic halo have to be treated theoretically.
A model with realistic source and gas distributions taking into account spallation for different nuclei shall be considered in the future. Here we deal with the spatial diffusion and stochastic reacceleration of protons with different primary injection functions. The production of the initial spectra could occur in supernova remnants by shock wave acceleration (for a review see Kirk (2000)).
The diffusion mechanism is provided by the scattering of the particles through plasma waves superimposed on the Galactic background magnetic field. In the quasilinear approximation of small electromagnetic pertubations the transport equation for the isotropic phase space distribution function, $f(t, \boldsymbol{x}, p)$, is obtained with only one momentum coordinate p after averaging over pitch angle and the gyrophase of the particles (e.g. Schlickeiser (2001)).

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1.1 The stationary transport equation

In this work we will assume a cylindrical geometry of the Galactic disk (height z_0, radius r_0) and the halo (height H, radius L) introducing the two remaining space coordinates r and z. The stationary transport equation without convection and without continuous loss terms like synchrotron radiation, which can be neglected for nuclei, then reads:

$$\left[\kappa_0\kappa(p)\left(\frac{1}{r}\partial_r r\partial_r + \partial_z^2\right) + \frac{1}{p^2}\partial_p\left(p^2 a_2(p)\partial_p\right) - \frac{1}{T_c}\right] f(r,z,p) = -S(r,z,p)\ . \quad (1)$$

The first and second term describe the spatial and the momentum diffusion with the diffusion coefficients $\kappa_0\kappa(p)$ and $a_2(p)$, respectively, both assumed independent of position here. The third term describes the catastrophic losses of hadronic cosmic rays by spallation or radioactive decay by a time scale T_c. On the right hand side enters the source function, representing the preaccelerated initial cosmic ray distribution in space and momentum.
The diffusion coefficients can be determined by quasilinear test particle calculations for arbitrary plasma modes. For a mixture of slab Alfvén waves and fast magnetosonic waves propagating along the background magnetic field we use the formulas of Schlickeiser (2001) for K_1 and D_1

$$\kappa_0\kappa(p) = K_1 p^{2-q} \quad (2)$$
$$D(p) = D_1 p^q \quad (3)$$

and a wave number spectrum $I(k) = I_0 k^{-q}$ with spectral index q. We define additionally $\eta := 2 - q$.

1.2 The solution

If the source function and the diffusion coefficients are separable, the differential equation can be solved by applying the "scattering time" method (Wang and Schickeiser, 1987) under "free-escape" boundary conditions (Lerche and Schlickeiser, 1985) by a series ansatz of the form

$$f(r,z,x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn} t_{mn}(r,z)\Phi_{mn}(x) \tag{4}$$

wherein $x := \frac{p}{m_p c}$. The eigenvalues λ_{mn} of the spatial problem enter by the variable

$$\begin{aligned}\phi_{mn} &:= \frac{T_f}{T_{mn}(\lambda_{mn})} \\ &= (m_p c)^{2(2-q)}\frac{K_1}{D_1}\left(\frac{y_n^2}{L^2} + \frac{(2m-1)^2\pi^2}{4H^2}\right)\end{aligned} \tag{5}$$

in the separated momentum equation (Appendix A) which is of confluent hypergeometric type. Here T_{mn} and T_f are the time scales for the spatial and the momentum diffusion for protons at 1 GeV. y_n are the zeros of the Bessel function J_0. In the general case the solution consists of the confluent hypergeometric functions M and U (Schlickeiser, 2001), in the case of no catastrophic losses for protons we get modified Bessel functions I and K (cf. Mause (1993)):

$$\Phi_{mn}(x) = \int_0^{\infty} dx_0 \left[x_0^2 T_f S(x_0)\right] G_{mn}(x,x_0) \tag{6}$$

with

$$G_{mn}(x,x_0) = \frac{1}{\eta} x^{\frac{\eta-3}{2}} x_0^{\frac{\eta-3}{2}} \tag{7}$$
$$\begin{cases} I_{\frac{3-\eta}{2\eta}}(\frac{\sqrt{\phi_{mn}}}{\eta}x^{\eta})K_{\frac{3-\eta}{2\eta}}(\frac{\sqrt{\phi_{mn}}}{\eta}x_0^{\eta}) & \text{for } x \le x_0 \\ I_{\frac{3-\eta}{2\eta}}(\frac{\sqrt{\phi_{mn}}}{\eta}x_0^{\eta})K_{\frac{3-\eta}{2\eta}}(\frac{\sqrt{\phi_{mn}}}{\eta}x^{\eta}) & \text{for } x_0 \le x \end{cases}.$$

The integrals can be calculated numerically for given spatial and momentum source functions $S(r,z)$ and $S(x)$ and in special cases analytically. As the solution consists of an infinite double sum over the eigenvalue indices m and n, practically it has be to cut off at a certain number of terms if the accuracy is sufficient.

2 Studies of the solution

2.1 Theoretical consistency checks of the momentum solution

1. Inserting the solution fulfills the differential equation (see Appendix A).
2. The solution for no catastrophic losses is also obtained if $T_c \to \infty$ in the M-U-solution.

2.2 The limit $q \to 2$

In the limit $q \to 2$ we get the following Green's function of the momentum solution for a certain eigenvalue (Schlickeiser, 2001):

$$G_{mn}(x,x_0) = \frac{(xx_0)^{-\frac{3}{2}}}{2\sqrt{\frac{9}{4}+\phi_{mn}}} \tag{8}$$
$$\begin{cases} (x/x_0)^{\sqrt{\frac{9}{4}+\phi_{mn}}} & \text{for } 0 \le x \le x_0 < \infty \\ (x/x_0)^{-\sqrt{\frac{9}{4}+\phi_{mn}}} & \text{for } 0 < x_0 \le x \le \infty \end{cases}.$$

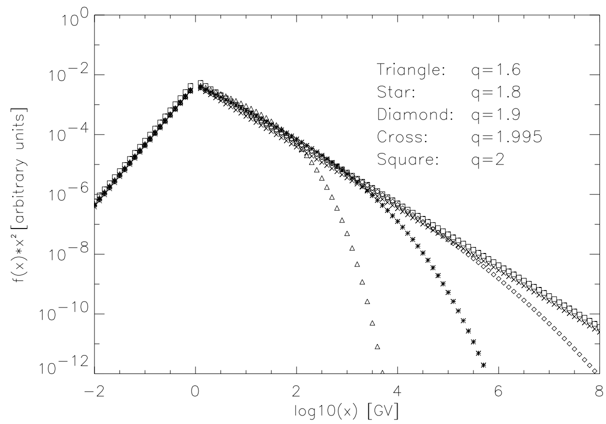

Fig. 1. Momentum solution for a δ-injection at 1 GV for different spectral indices of the plasma wave spectrum. The numerically calculated functions for $q < 2$ converge towards the limiting curve (8) for $q = 2$ for one special eigenvalue ϕ_{mn}.

The appropriate numerically calculated phase space distribution function for $q < 2$ converges towards this limiting curve for one special eigenvalue ϕ_{mn}, as it should be (see Figure 1).

2.3 Power law momentum injection function

Now we study the response of the momentum solution for an injection spectrum

$$S(x_0) = s_0 x_0^{-\beta-2} \tag{9}$$

covering the interval from x_{min} to x_{max} by varying the spectral index q of the plasma turbulence, the ratio T_f/T_0 of the momentum to the spatial diffusion time scale and the spectral index β ot the initial spectrum.

A further test of the analytical solution and especially of the numerical code are the following approximations (see Appendix B) for the momentum solution

$$\Phi_{mn}(x) \simeq \frac{s_0 T_f}{\beta-1}\,\frac{1}{3-\eta}\, x_{min}^{1-\beta} x^{\eta-3} \tag{10}$$

for $x \ll x_c$ and

$$\Phi_{mn}(x) \simeq s_0 T_f \frac{1}{\phi_{mn}} x^{-\eta-\beta-2} \tag{11}$$

for $x \gg x_c$ according to the characteristic momentum

$$x_c := \left(\frac{\eta}{\sqrt{\phi_{mn}}}\right)^{\frac{1}{\eta}} \tag{12}$$

which emerges in the argument of the Bessel function.
In the first case the influence of the source function disappeared and momentum diffusion dominates and in the second case the initial power law is only steepened by $q-2$ and momentum diffusion plays a minor role.
Above the maximum injection momentum x_{max} all spectra

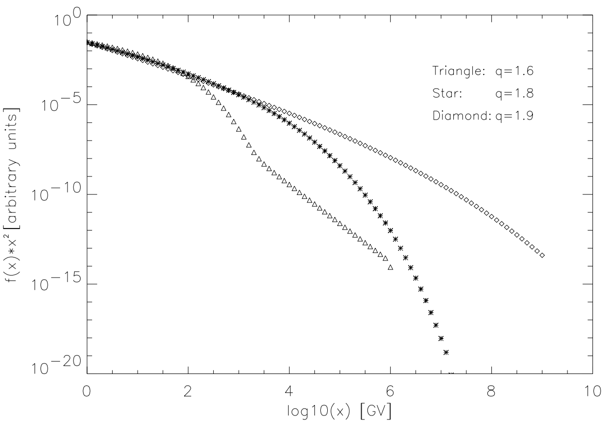

Fig. 2. Phase space distribution function for a power law momentum source function (9) with spectral index $\beta = 1.75$ extending to $x_{max} = 10^6$ for different spectral indices q of the plasma turbulence spectrum and $\phi = \frac{T_f}{T_0} = \frac{1}{10}$. The three upper flat dotted lines are the power law approximations (10) for small x with the mentioned parameters β and ϕ and the quoted q-values. The lower steeper dotted line is the respective approximation (11) for large x for the case $q = 1.6$.

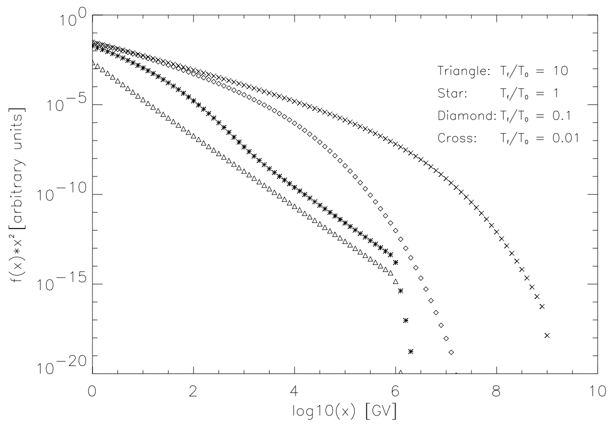

Fig. 3. Phase space distribution function with an injection power law (9) for $q = 1.8$ and different ratios of the momentum to the spatial diffusion time scale $\frac{T_f}{T_0}$. The upper flat dotted curve is the approximation (10) for small x, the two lower steeper curves are the approximations (11) for large x for the cases $\frac{T_f}{T_0} = 10$ and $\frac{T_f}{T_0} = 1$.

cut off exponentially, independent of the source index β (see Appendix B):

$$\Phi_{mn}(x) \sim x^{-\frac{3}{2}} e^{-\left(\frac{x}{x_c}\right)^{\eta}} \; . \qquad (13)$$

The approximating curves from Equations (10) and (11) are superimposed in the Figures 2, 3 and 4.

The strong effect of the wave number distribution of the turbulence spectrum shows Figure 2. For larger values of q the momentum diffusion process can accelerate particles to higher momenta.
A similar consequence has the increase of T_f/T_0 (see Figure 3). If the time scale for momentum diffusion in comparison to the spatial diffusion is smaller, more particles can reach higher momenta.
From Figure 4 one notices that a flatter injection spectrum of course results in a flatter processed spectrum. If momentum acceleration dominates, the influence of the primary distribution vanishes for large momenta.

2.4 Source spectrum with a dispersive index

Assuming a superposition of source spectra with different spectral indices, like it can occur if the sources are supernova remnants producing power laws with variable steepness, we insert the dispersive source function

$$S(x_0) = s_0 x_0^{-\beta - 2 + \frac{1}{2}\sigma^2 \, \ln\left(\frac{x_0}{x_r}\right)} \qquad (14)$$

with a mean index $< \beta >$ and a dispersion parameter σ at a reference momentum x_r.
A comparison to the single power law distribution is shown in Figure 5. A typical dispersion parameter is in the order of $\sigma = 0.25$ (Büsching et al., 2001). To see an effect, here we take an extreme value of $\sigma = 0.55$. As expected, the flattest source spectra of the sample dominate for large values thereby shifting the particles to higher momenta.

3 Summary

We have investigated the transport equation for Galactic cosmic rays including stochastic reacceleration. Here we present analytical solutions in the form of eigenfunction expansions. Besides some theoretical consistency checks of the momentum solution, we studied the spectra provided by numerically calculating the integrals of the momentum solution functions for different parameter sets.
The numerical code seems to be consistent with the obtained analytical approximations. The influence of momentum diffusion becomes stronger for steeper spectra of the plasma wave turbulence in the interstellar medium and of course, if the ratio of the momentum to the spatial diffusion time scale decreases.
In the future we will examine more terms of the sum of the eigenfunctions and discuss the spatial distribution functions. In the end we want to model the spectra of different nuclei by taking into account spallation processes.

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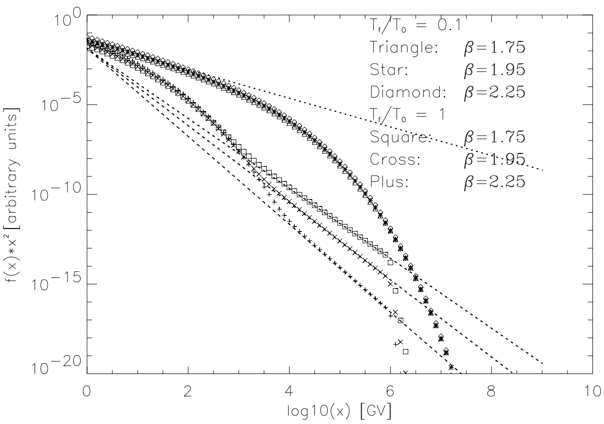

Fig. 4. Phase space distribution function for $q = 1.8$ and a source function (9), but with varying β. The curves are calculated for two different time scale ratios. Again the upper dotted curve is yielded with equation (10) for small x and $\frac{T_f}{T_0} = 0.1$ and the lower steeper curves (11) approximate the three cases for $\frac{T_f}{T_0} = 1$ for large x.

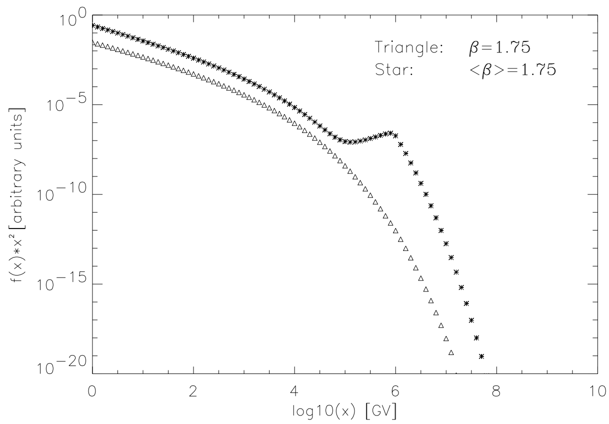

Fig. 5. Source distribution with dispersion as in Equation (14) with $\sigma = 0.55$ and $< \beta >= 1.75$. Additionally drawn is the result for a single power law (9) with $\beta = 1.75$.

Appendix A The momentum equation and inserting its solution

The momentum differential equation for the Green's function reads:

$$\left[\frac{d}{dx}\left(x^{4-\eta}\frac{d}{dx}\right) - \phi x^{2+\eta}\right] G(x, x_0) = -\delta(x - x_0)\ . \quad \text{(A1)}$$

Here $\phi \equiv \phi_{mn}$.

Inserting the expression for the Green's function (7) on the left hand side, transforming to the new variable z through $x = \left(\frac{\eta z}{\sqrt{\phi}}\right)^{\frac{1}{\eta}}$ and using the relations for the derivatives of the modified Bessel functions (Abramowitz and Stegun, 1984), e.g.

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k [z^\nu I_\nu(z)] = z^{\nu-k} I_{\nu-k}(z) \quad \text{(A2)}$$

after some calculations we obtain the δ-function on the right hand side.

Appendix B Approximations of the momentum solution for a power law

Plugging in the injection power law (9) and introducing the variable (12) we obtain for the momentum function

$$\Phi_{mn}(x) = s_0\, T_f\, \frac{1}{\eta^2} x^{\eta-\beta-2} \quad \text{(B1)}$$
$$\left[K_\nu(B) \int_{\left(\frac{x_{min}}{x}\right)^\eta}^{1} dy\ y^{-\frac{\eta+2\beta+1}{2\eta}} I_\nu(By) + I_\nu(B) \int_1^{\left(\frac{x_{max}}{x}\right)^\eta} dy\ y^{-\frac{\eta+2\beta+1}{2\eta}} K_\nu(By)\right]$$

with

$$B := B(x) = \frac{\sqrt{\phi_{mn}} x^\eta}{\eta} = \left(\frac{x}{x_c}\right)^\eta \text{ and } \nu := \frac{3-\eta}{2\eta}\ . \quad \text{(B2)}$$

Applying the approximation formulas of the modified Bessel function for small and large arguments

$$I_\nu(z) \simeq \begin{cases} \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)} & \text{for } |z| \ll 1 \\ \frac{e^z}{\sqrt{2\pi\, z}} & \text{for } |z| \gg 1 \end{cases} \quad \text{(B3)}$$

$$K_\nu(z) \simeq \begin{cases} \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu} & \text{for } |z| \ll 1 \\ \sqrt{\frac{\pi}{2z}} e^{-z} & \text{for } |z| \gg 1 \end{cases} \quad \text{(B4)}$$

after some additionally approximations we get the formulas (10) and (11).

In the case of $x > x_{max}$ only an integral independent of x remains:

$$\Phi_{mn}(x) = s_0 T_f \frac{1}{\eta} x^{\frac{\eta-3}{2}} K_\nu\left(\frac{x}{x_c}\right)^\eta \quad \text{(B5)}$$
$$\int_{x_{min}}^{x_{max}} dx_0\ x_0^{-\beta+\frac{\eta-3}{2}} I_\nu\left(\frac{x_0}{x_c}\right)^\eta\ .$$

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