Mathematical Illustration of the Gibbs Phenomenon

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1 Motivation

Recall the claim that the Gibbs phenomenon occurs near points of discontinuity of a periodically extended function $f_p(x)$ that is approximated by a Fourier series in which only a *finite* number of terms are kept. Near a point of discontinuity, the Fourier series approximation oscillates about the numerical value it should achieve according to the Fourier convergence theorem, which is valid in the infinite series limit. Further, the overshoot near the discontinuity does not vanish as more and more modes are retained, contrary to intuition. Instead, the overshoot is finite no matter what finite number of modes N are retained, even though the region of overshoot gets progressively smaller as $N \to \infty$.

2 Model Problem

A simple constant function illustrates the Gibbs phenomenon nicely. Consider the function,

$$f(x) = \begin{cases} 1 \; , & 0 \le x < \pi \\ 0 \; , & \pi \le x < 2\pi \end{cases} \tag{1}$$

which in its infinite periodic extension (of period 2π) has discontinuities at $x = n\pi$ for any integer n. Now define the complex Fourier series approximation to $f(x)$:

$$f_N(x) \equiv \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} f_k e^{ikx} \; . \tag{2}$$

Note that the sum in equation (2) ranges over a total of $N+1$ modes, including $k = 0$. Using an inner product appropriate for a linear vector space of 2π-periodic complex functions,

$$\langle g(x), h(x) \rangle = \int_0^{2\pi} \overline{g(x)} h(x) dx \; , \tag{3}$$

where the overbar denotes complex conjugation, we find that the Fourier coefficients f_k are given by

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \; . \tag{4}$$

Upon inserting expression (4) into (2), we may rewrite the Fourier approximation to $f(x)$ as

$$f_N(x) = \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{|k| \le \frac{N}{2}} e^{ik(x-y)} \right] f(y) dy \equiv \frac{1}{2\pi} \int_0^{2\pi} D_N(x-y) f(y) dy \; . \tag{5}$$

The function

$$D_N(z) = \sum_{|k| \le \frac{N}{2}} e^{ikz} = 1 + 2\sum_{k=1}^{\frac{N}{2}} \cos kz \tag{6}$$

is known as the *Dirichlet kernel,* an analysis of which reveals the oscillatory behavior associated with the Gibbs phenomenon.

There is a trick that helps us to simplify the expression for $D_N(z)$. Notice that

$$(1-e^{iz})D_N(z) = (1-e^{iz})\sum_{|k|\le \frac{N}{2}} e^{ikz} = \sum_{|k|\le\frac{N}{2}} e^{ikz} - e^{i(k+1)z} = e^{-i\frac{Nz}{2}} - e^{i(\frac{N}{2}+1)z} \ ,$$

which upon further massaging gives

$$(1-e^{iz})D_N(z) = e^{i\frac{z}{2}}\left(e^{-i(\frac{N+1}{2})z} - e^{i(\frac{N+1}{2})z}\right) \ .$$

We then have

$$D_N(z) = \frac{e^{-i(\frac{N+1}{2})z} - e^{i(\frac{N+1}{2})z}}{e^{-i\frac{z}{2}} - e^{i\frac{z}{2}}} = \begin{cases} \frac{\sin(\frac{N+1}{2}z)}{\sin\frac{z}{2}} & , \quad z \ne 2j\pi \\ N+1 & , \quad z = 2j\pi \end{cases} \ . \tag{7}$$

A plot of the Dirichlet kernel is shown in figure 1 for various values of N. The main thing you should get out

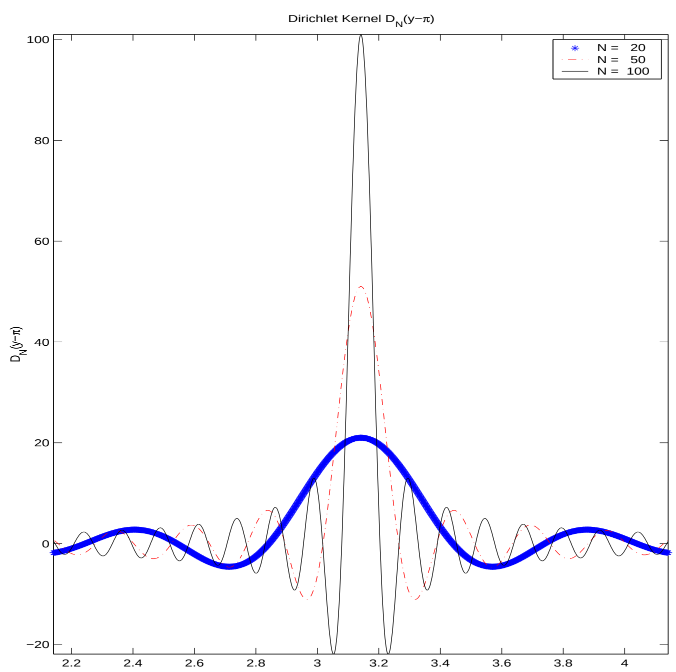

Figure 1: Dirichlet Kernel $D_N(y-\pi)$

of figure 1 is that the Dirichlet kernel becomes an increasingly better approximation of a Dirac delta function in the limit $N \to \infty$, which by equation (5) means that $f_N(x) \to f(x)$ in the large N limit. However, the tails of D_N oscillate rapidly. This fact accounts for the oscillations in the Fourier series approximation (5). To see this more clearly, let us calculate the integral in (5) explicitly. Using the definition (1) for $f(x)$, we may write

$$f_N(x) = \frac{1}{2\pi}\int_0^{\pi}(1)D_N(x-y)dy = \frac{1}{2\pi}\int_{x-\pi}^{x} D_N(z)dz \ , \tag{8}$$

where we used the variable $z = x - y$ in the final integral of equation (8). It is then helpful to split this integral into three separate pieces as shown below:

$$f_N(x) = \frac{1}{2\pi}\left[\int_0^x D_N(z)dz + \int_{-\pi}^{0} D_N(z)dz + \int_{x-\pi}^{-\pi} D_N(z)dz\right] \ . \tag{9}$$

Now we need to note the following facts regarding the Dirichlet kernel.

1. $\frac{1}{2\pi}\int_0^{2\pi} D_N(z)dz = 1$ (plug in the function $f_N(x) = f(x) = 1$ into (5) to see this)
2. $D_N(z)$ is by definition an even function, so $\frac{1}{2\pi}\int_{-\pi}^{0} D_N(z)dz = \frac{1}{2}$ (using the result above)
3. As $N \to \infty$, $D_N(z)$ vanishes on every interval that does not contain a singular point $z = 2j\pi$ for integer j. To be more technical, this means that for any given $\delta > 0$ and $\epsilon > 0$, there exists an integer $N(\delta,\epsilon) > 0$ such that $|D_N(z)| < \epsilon$ provided $N > N(\epsilon,\delta)$ and $\delta \le z \le 2\pi - \delta$. We do not prove this fact here, but it plays an important role in approximating two of the integrals in expression (9).

It is sufficient for the illustration of the Gibbs phenomenon to focus our attention on the singular point occurring at $x = 0$. The Fourier convergence theorem guarantees that $f_N(x) = 1$ for x arbitrarily close to 0 (but not equal to 0), provided we consider the infinite summation of Fourier modes. We will see that the *finite truncation* gives us something different near the singular point at $x = 0$.

To aid your intuition, I have plotted in figure 2 the integral $\int_0^x D_N(y-\pi)dy$ as a function of x with $N = 100$ modes. As x ranges from 0 to 2π, it is clear that the integral is non-vanishing (approximately) only after x passes through the singularity at $x = \pi$. That is, the integral is non-vanishing provided the integration range contains a point for which the argument of $D_N(y-\pi)$ equals an integer multiple of 2π, which is at $y = \pi$. Thus, it should be clear that $\int_{x-\pi}^{-\pi} D_N(y)dy$ is negligibly small as $N \to \infty$ provided x is not near π.

With these observations in hand, if x is comfortably removed from π we have to good approximation,

$$f_N(x) \simeq \frac{1}{2} + \frac{1}{2\pi}\int_0^x D_N(y)dy \ . \tag{10}$$

For x not near the singular point at $x = 0$, the integral in (10) is approximately $\frac{1}{2}$, since the majority of the contribution comes from the region near $x = 0$ and we may replace the upper bound of integration x by π (this is validated by the theory of asymptotic approximations of integrals, which you may read about in the classic book "Advanced Methods for Scientists and Engineers" by Bender and Orszag). We therefore have as $N \to \infty$

$$f_N(x) \simeq 1 \quad , \quad 0 \ll x \ll \pi \ , \tag{11}$$

which is precisely the result we expect from the Fourier convergence theorem away from points of singularity. Also, note again that the oscillatory nature of the integral (see figure 2) is the reason why the Fourier approximation oscillates about the value to which it converges.

Now consider points near (but not precisely equal to) the singularity at $x = 0$. A crucial thing to notice from figure 2 is that $\int_0^x D_N(y)dy$ has alternating maxima and minima wherever D_N vanishes, namely at the points $x_j = \frac{2j\pi}{N+1}$ for any integer j. For the interval of concern to us here, the maximum value of this integral near $x = 0$ occurs at the point $x_1 = \frac{2\pi}{N+1}$, which approaches 0 as $N \to \infty$. So, for large enough N we may approximate $f_N(x)$ near $x = 0$ by

$$f_N(x) \simeq \frac{1}{2} + \frac{1}{2\pi} \int_0^{\frac{2\pi}{N+1}} D_N(y)dy = \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin z}{z} dz \ , \tag{12}$$

where in the last step we have used the substitution $z = \frac{N+1}{2} y$ and equation (7). The final integral is a well-known one that you may deduce using Cauchy's theorem or a table of integrals:

$$\frac{1}{\pi} \int_0^{\pi} \frac{\sin z}{z} dz \simeq .58949 \ . \tag{13}$$

Thus, near the singularity at $x = 0$ we have

$$f_N(x) \simeq 1.08949 \ , \tag{14}$$

which does NOT approach unity in the limit $N \to \infty$. This is the Gibbs phenomenon. It is purely a product of the finite truncation. You will often see the statement (i.e. Haberman's book) that the overshoot is approximately 9% of the jump discontinuity. Hopefully now you see why this is so. It is due to the integral (13) that naturally appears in the numerical evaluation of a finitely-truncated Fourier series near a point of discontinuity.

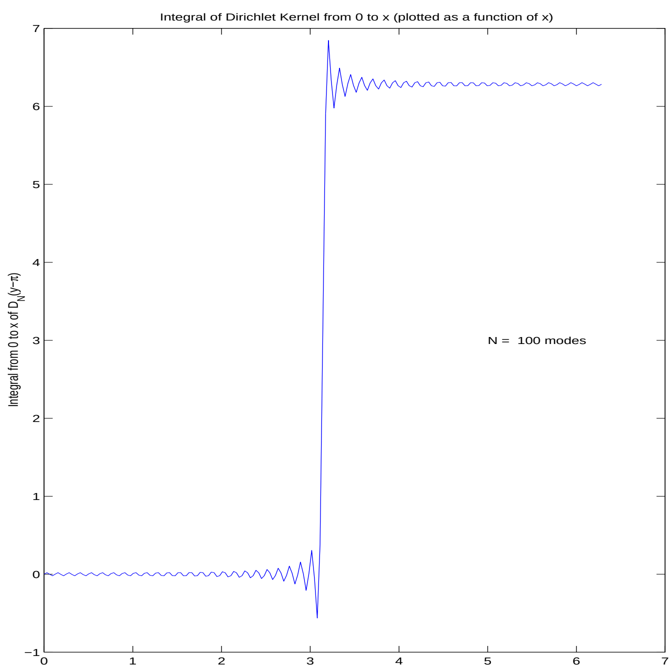

Figure 2: $\int_0^x D_N(y-\pi)dy$ for $N = 100$ modes.