Signal Transmission

Contents

1 Wave Equation in transmission lines

A transmission line is composed of 2 conductors isolated by dielectric medium with dielectric constant ϵ:

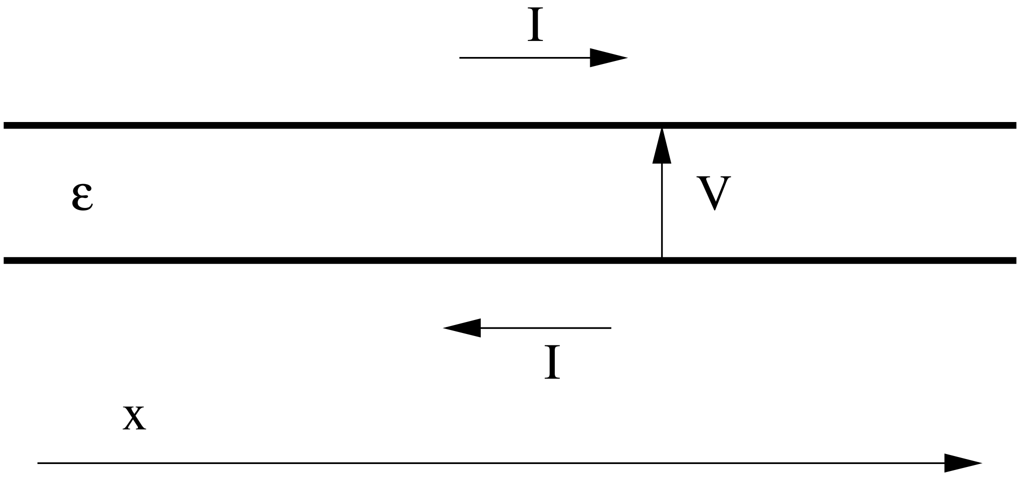


Figure 1: Transmission line

The voltage between both lines (V) and the current are function both of position and time $V(x,t)$ $I(x,t)$

1.1 Equivalent circuit

- The conductors are isoleted by a dielectric (or air), so there will be a capacitance between the conductors.

- If there is a current in the lines, then a magnetic field $\vec{B}$ is generated producing a magnetic flux ϕ. If the current changes then $\vec{B}$ and also ϕ, generating an inducted voltage and an autoinduction L

Then we can represent our transmission line as a serie of inductors and capacitors as showed in figure **??** where:

$L = L_l \cdot \Delta x$ L_l autoinductance per unit length
$C = C_l \cdot \Delta x$ C_l capacitance per unit length

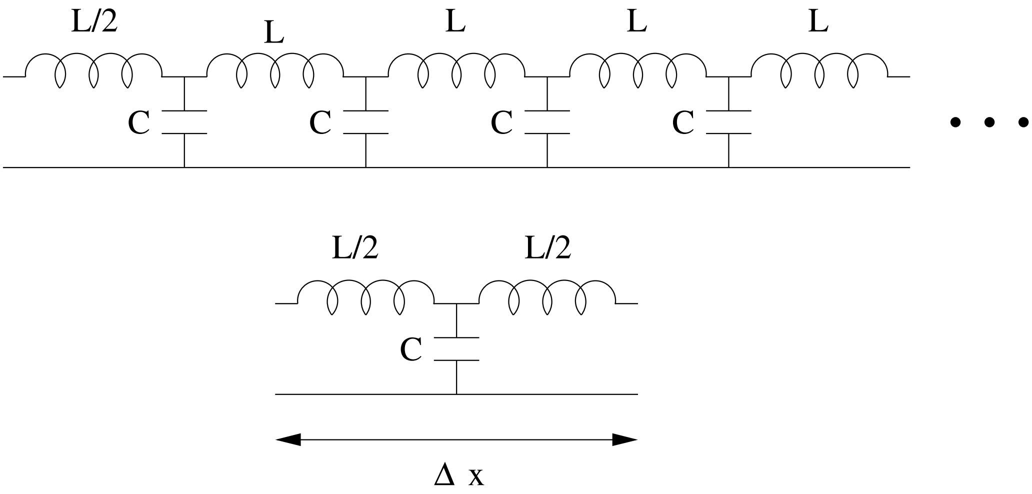


Figure 2: Equivalent circuit

We can compute the impedance of such series (infinite) just taking into account that in any interval ΔX is equivalent to the circuit sketched in the lower part of figure **??** finding that:

$$Z_0 = Z_{L/2} + (Z_{L/2}||Z_C) = \sqrt{\frac{L}{C}}\sqrt{1-\frac{w^2LC}{4}} = \sqrt{\frac{L_l}{C_l}}\sqrt{1-\frac{w^2L_lC_l(\Delta x)^2}{4}}$$

Making $\Delta x \rightarrow 0$ in order to represent a continum line we have that the characteristic impedance of an ideal transmission line is:

$$Z_0 = \sqrt{\frac{L_l}{C_l}}$$

1.2 Example 1: Strip line

This is the case that we usually find in printed circuits: a) Capacity:

$$C = \varepsilon\frac{Surface}{distance} = C_l \cdot \Delta x = \varepsilon\frac{b \cdot \Delta x}{a} = \varepsilon\frac{b}{a}\Delta x$$

$$C_l = \varepsilon\frac{b}{a}$$

b) Self-inductance:

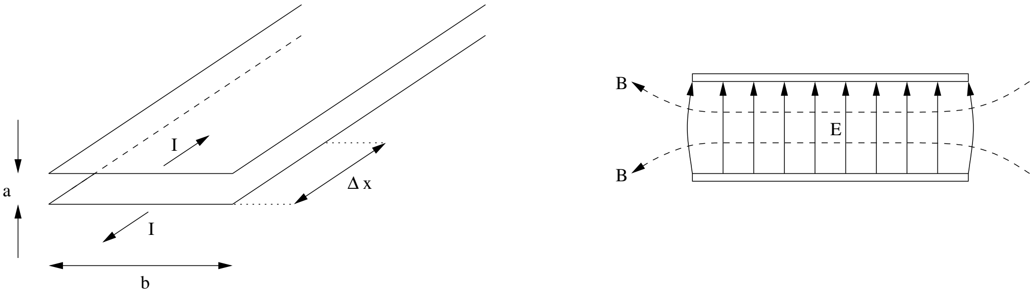


Figure 3: Strip line: geometry and fields

- 1 strip

$$I = \oint \vec{H} d\vec{l} = H \cdot 2b \rightarrow B = \frac{\mu I}{2b}$$

- 2 strips: B is added inside and substracted (annule) outside.

$$B = \frac{\mu I}{b}$$

Figure 4: Strip line: magnetic field for 1 strip and 2 strips

$$\phi = \vec{B} \cdot \vec{S} = \frac{\mu I}{b} a \cdot \Delta x = \mu \frac{a}{b} \Delta x I$$

$$|V_{ind}| = \frac{d\phi}{dt} = \mu \frac{a}{b} \Delta s \frac{dI}{dt} = L \frac{dI}{dt}$$

$$L = \mu \frac{a}{b} \Delta x \rightarrow L_l = \mu \frac{a}{b}$$

c) Characteristic impedance:

$$Z_0 = \sqrt{\frac{L_l}{C_l}} = \sqrt{\frac{\mu}{\varepsilon}} \frac{a}{b} == \sqrt{\frac{\mu_r}{\varepsilon_r}} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{a}{b}$$

$$Z_0 = \frac{377\Omega}{sqrt \varepsilon_r} \frac{a}{b} \quad \mu_r = 1$$

1.3 Example 2: Line pair, not blinded

$$Z_0 = \frac{120\Omega}{\sqrt{\epsilon_r}} \ln\left(\frac{D}{r}\right)$$

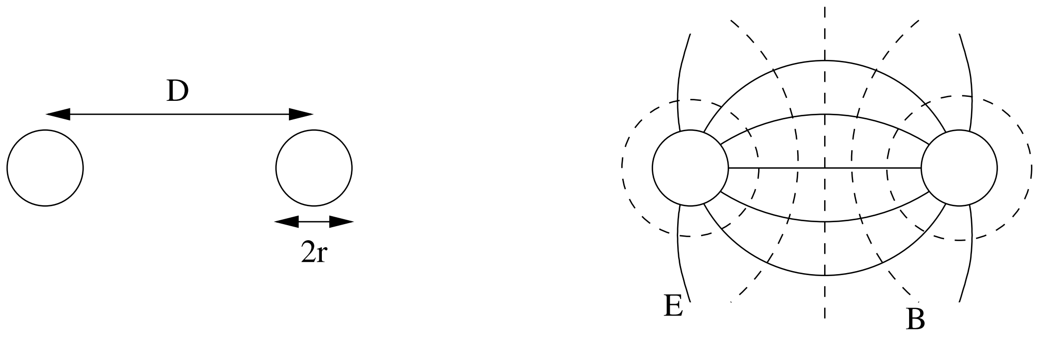
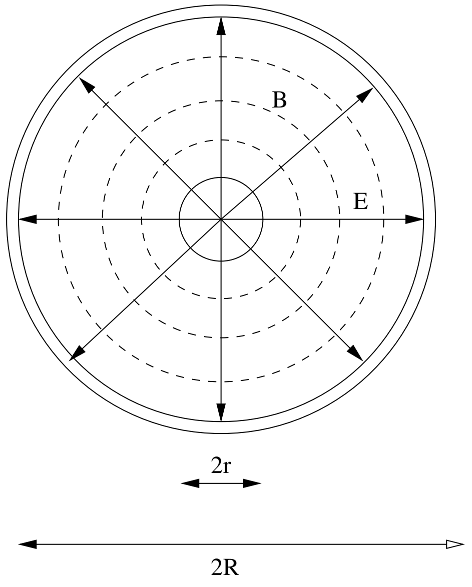


Figure 5: Line pair: geometry and fields

1.4 Example 3: Coaxial cable



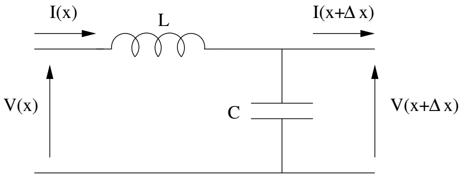
$$C_l = 2\pi \frac{\varepsilon}{\ln(R/r)}$$

$$L_l = \frac{\mu}{2\pi} \ln\left(\frac{R}{r}\right)$$

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}} \frac{\ln(R/r)}{2\pi} = \frac{60\Omega}{\sqrt{\varepsilon_r}} \ln\left(\frac{R}{r}\right)$$

2 Wave equation

Let’s consider a transmision line element of length Δx:



$$L = L_l \cdot \Delta x$$

$$C = C_l \cdot \Delta x$$

a) Intensity Kirchoff law

$$I(x) = C\frac{dV}{dt} + I(x+\Delta x) \rightarrow I(x+\Delta x) - I(x) = -C_l\Delta x\frac{dV}{dt}$$

$$\frac{I(x+\Delta x) - I(x)}{\Delta x} = -C_l\frac{dV}{dt}$$

$$\frac{dI}{dx} = -C_l\frac{dV}{dt}$$

b) Voltage Kirchoff law

$$V(x) = L\frac{dI}{dt} + V(x+\Delta x) \rightarrow V(x+\Delta x) - V(x) = -L_l\Delta x\frac{dI}{dt}$$

$$\frac{V(x+\Delta x) - V(x)}{\Delta x} = -L_l\Delta x\frac{dI}{dt}$$

$$\frac{dV}{dx} = -L_l\frac{dI}{dt}$$

$$\frac{d}{dx}a) \rightarrow \frac{d^2I}{dx^2} = -C_l\frac{d^2V}{dxdt}$$

$$\frac{d}{dt}b) \rightarrow \frac{d^2V}{dtdx} = -L_l\frac{d^2I}{dt^2}$$

$$\frac{d^2I}{dx^2} = L_lC_l\frac{d^2I}{dt^2}$$

$$\frac{d}{dt}a) \rightarrow \frac{d^2I}{dxdt} = -C_l\frac{d^2V}{dt^2}$$

$$\frac{d}{dx}b) \rightarrow \frac{d^2V}{dt^2} = -L_l\frac{d^2I}{dxdt}$$

$$\frac{d^2V}{dx^2} = L_lC_l\frac{d^2V}{dt^2}$$

These equations have the same structure as wave equations:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

The solutions are periodic, $V(x \pm vt)$, $I(x \pm vt)$ where v is the propagation velocity, that in the previous equations is

$$v = \frac{1}{\sqrt{L_lC_l}}$$

2.1 Relation between V and I. Characteristic impedance

$V(x,t) = V(x-vt)$ and $I(x,t) = I(x-vt)$ are solutions of the wave equation, but also should satisfy the realtions a) and b):

$$\frac{dV}{dx} = \frac{dV}{du}\frac{du}{dx} = fracdVdu \quad (u = x - vt)$$

$$\frac{dI}{dt} = \frac{dI}{du}\frac{du}{dt} = -v\frac{dI}{du}$$

Taking equation b), and substituting we obtain:

$$\frac{dV}{dx} = -L_l\frac{dI}{dt} \rightarrow \frac{dV}{du} = -L_l(-v)\frac{dI}{du} = \sqrt{\frac{L_l}{C_l}}\frac{dI}{du}$$

$$dV = \sqrt{\frac{L_l}{C_l}}dI \rightarrow V(x-vt) = Z_0 I(x-vt)$$

The value Z_0, by comparing with Ohm's law is called characteristic impedance.

$$Z_0 = \sqrt{\frac{L_l}{C_l}}$$

2.2 Propagation velocity

$$v = \frac{1}{L_l C_l}$$

In the geometries of the previous examples it can be verified that $L_l C_l = \mu\varepsilon$ and this relation is true for all geometries. This means that

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

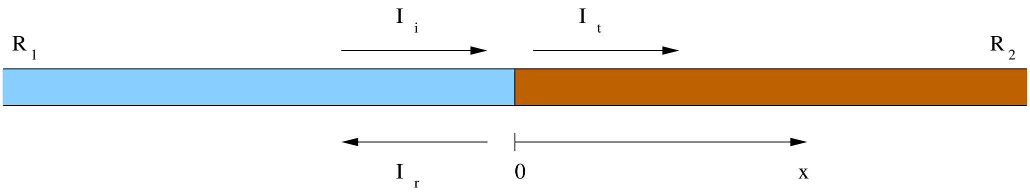
The signal transmission is that of the electromagnetic wave that travels in the conductors. Usually $\mu_r = 1$, so,

$$v = \frac{C}{\sqrt{\varepsilon_r}}$$

and the velocity depends only on the dielectric.

2.3 Reflexions. Impedance adaptation

Let's study the case of a signal that pass from a line of impedance R_1 to another of impedance R_2. Due to continuity there will be a reflexion:



Incident signal: $V_i(x-vt)$; $I_i(x-vt) \rightarrow V_i = R_1 I_i$

Transmited signal: $V_t(x-vt)$; $I_t(x-vt) \rightarrow V_t = R_2 I_t$

Reflected signal: $V_r(x-vt)$; $I_r(x-vt) \rightarrow V_r = R_1 I_r$

Applying Kirchoff’s laws at point 0:

$$I_i - I_r = I_t \rightarrow \frac{V_t}{R_2} = \frac{V_i}{R_1} - \frac{V_r}{R_1} \rightarrow V_t = \frac{R_2}{R_1}(V_i - V_r)$$

$$V_i + V_r = V_t$$

Combining both relations we have

$$V_t = V_i + V_r = \frac{R_2}{R_1}(V_i - V_r)$$

$$V_r = \frac{R_2 - R_1}{R_2 + R_1} V_i = \rho V_i$$

$$V_t = \frac{2R_2}{R_2 + R_1} V_i = (1+\rho) V_i$$

$$\rho = \frac{R_2 - R_1}{R_2 + R_1}$$

ρ is known as reflection coefficient.
We can consider three cases:

$R_2 = R_1$	adaptation	$\rho = 0$	$V_t = V_i$	$V_r = 0$
$R_2 = 0$	short-circuit	$\rho = -1$	$V_t = 0$	$V_r = -V_i 0$
$R_2 = \infty$	open-circuit	$\rho = +1$	$V_t = 2V_i$	$V_r = V_i$