Lagrangian Wave Mixing in Cosmic Ray Modified Flows

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Abstract

In this paper, a Lagrangian approach to wave mixing in cosmic-ray modified flows is developed. In particular, the wave mixing equations for short wavelength sound waves and entropy waves in a large scale background flow obtained by Webb et al. (1997) are cast in the form of two coupled equations for the Lagrangian fluid displacement ξ and the Lagrangian entropy perturbation ΔS. The fluid displacement vector $\xi(x,t)$ satisfies a second order wave equation, which describes the propagation of short wavelength sound waves in the large scale background flow, with source terms owing to the the cosmic ray squeezing instability, and the entropy perturbation ΔS. The source of the squeezing instability is identified with the non-adiabatic compression of the fluid element by the cosmic rays. The entropy perturbation ΔS, is advected with the background flow. It is noted that these equations can be generalised to arbitrary wavelength waves.

1 Introduction

Webb, Brio, Zank and Story (1997) derived linear and weakly nonlinear evolution equations describing wave coupling and cosmic ray squeezing instabilities for short wavelength sound waves and entropy waves in a large scale background flow. The equations were used in Webb et al. (1999) to discuss instabilities in cosmic ray modified shocks with application to SNR shocks. The wave interaction equations were written in terms of the Eulerian density perturbations for the waves. In this paper, the equivalent Lagrangian form of the equations is obtained.

2 The Model

For the case of one dimensional flow along the x-axis, the two-fluid cosmic ray hydrodynamical equations may be written in the form:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+p_g+p_c)=0, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho S)+\frac{\partial}{\partial x}(\rho u S)=0. \tag{3}$$

$$\frac{\partial p_c}{\partial t}+u\frac{\partial p_c}{\partial x}+\gamma_c p_c\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}\left(\kappa\frac{\partial p_c}{\partial x}\right)=0. \tag{4}$$

where $S=C_v\ln(p_g/\rho^{\gamma_g})$ is the gas entropy. In the above equations ρ, u, p_g and γ_g denote the density, fluid velocity, pressure and adiabatic index of the thermal gas; and p_c, γ_c and κ denote the cosmic ray pressure, adiabatic index and hydrodynamical diffusion coefficient respectively. C_v is the gas specific heat at constant volume.

3 Eulerian Wave Mixing Equations

From Webb et al. (1997,1999), linear, short wavelength waves propagating through a non-uniform, large scale background flow governed by (1)-(4), satisfy wave mixing equations of the form:

$$\frac{\partial \rho_i}{\partial t}+\frac{\partial}{\partial x}\left(\lambda_i\rho_i\right)+\sum_{j=1}^{3}\Lambda_{ij}\rho_j=0,\quad i=1,2,3, \tag{5}$$

where the $\{\rho_i : i = 1, 2, 3\}$ are the Eulerian density perturbations for the backward sound wave (ρ_1), entropy wave (ρ_2), and forward sound wave (ρ_3) respectively. The phase speeds of the waves are:

$$\lambda_1 = u - a_g, \quad \lambda_2 = u, \quad \lambda_3 = u + a_g, \tag{6}$$

where $a_g = (\gamma_g p_g/\rho)^{\frac{1}{2}}$ is the thermal gas sound speed. From the eigenvector relations the $\{\rho_j : j = 1, 2, 3\}$ are given by the equations:

$$\rho_1 = \frac{1}{2}\left(\delta\rho + \frac{\rho\delta S}{\gamma_g C_v} - \frac{\rho\delta u}{a_g}\right), \quad \rho_2 = -\frac{\rho\delta S}{\gamma_g C_v}, \quad \rho_3 = \frac{1}{2}\left(\delta\rho + \frac{\rho\delta S}{\gamma_g C_v} + \frac{\rho\delta u}{a_g}\right). \tag{7}$$

where $\delta\rho$, δu and δS are the total Eulerian density, fluid velocity and entropy perturbations respectively.

The wave interaction coefficients Λ_{ij} in (5) depend on the gradients of the non-uniform background flow and have the form:

$$\begin{aligned}
\Lambda_{11} &= \frac{1}{2}\left(\frac{(3-\gamma_g)}{2}R_x^+ - \frac{a_g S_x}{(\gamma_g - 1)C_v} + \frac{a_c^2}{\kappa} + \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x}\right), \\
\Lambda_{12} &= \frac{1}{2}\left(\frac{R_x^+ - R_x^-}{2} - \frac{a_g S_x}{\gamma_g(\gamma_g - 1)C_v} + \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x}\right) \equiv -\frac{1}{2a_g}\frac{du}{dt} + \frac{\zeta}{2\rho a_g}\frac{\partial p_c}{\partial x}, \\
\Lambda_{13} &= \frac{1}{2}\left(\frac{\gamma_g - 3}{2}R_x^- + \frac{(\gamma_g - 2)a_g S_x}{\gamma_g(\gamma_g - 1)C_v} + \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x} - \frac{a_c^2}{\kappa}\right), \\
\Lambda_{21} &= -\Lambda_{23} = \frac{a_g S_x}{\gamma_g C_v}, \quad \Lambda_{22} = 0, \\
\Lambda_{31} &= \frac{1}{2}\left(\frac{\gamma_g - 3}{2}R_x^+ - \frac{(\gamma_g - 2)a_g S_x}{\gamma_g(\gamma_g - 1)C_v} - \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x} - \frac{a_c^2}{\kappa}\right), \\
\Lambda_{32} &= -\Lambda_{12}, \\
\Lambda_{33} &= \frac{1}{2}\left(\frac{3-\gamma_g}{2}R_x^- + \frac{a_g S_x}{(\gamma_g - 1)C_v} + \frac{a_c^2}{\kappa} - \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x}\right),
\end{aligned} \tag{8}$$

where the parameter $\zeta = \partial \ln\kappa/\partial\ln\rho$; $du/dt = \partial u/\partial t + u\partial u/\partial x$ is the acceleration vector of the fluid; and $R^\pm = u \pm 2a_g/(\gamma_g - 1)$ are the Riemann invariants of isentropic gas dynamics. The $\partial p_c/\partial x$ terms in (8) correspond to the cosmic ray squeezing instability investigated by Drury and Falle (1986) and Zank and McKenzie (1987). The a_c^2/κ terms represent damping of the sound waves by the diffusing cosmic rays, and $a_c = (\gamma_c p_c/\rho)^{\frac{1}{2}}$ is the 'cosmic ray sound speed'. Note that the character of the instability depends on whether $\zeta < -1$, $\zeta = -1$ or $\zeta > -1$, where $\zeta = \partial\ln\kappa/\partial\ln\rho$. For quasi-parallel shocks in which the scattering wave field is due to a pre-existing Alfvén wave field upstream of the shock $\kappa \propto \rho^{-\frac{3}{2}}$, and $\zeta = -\frac{3}{2}$, whereas for a quasi-perpendicular shock in which $\kappa \sim \kappa_\perp$ is due to random walk of the field lines $\kappa \sim const.$, and $\zeta = 0$ (e.g., Webb et al. 1999).

4 Lagrangian Wave Interaction Equations

In this section we generalize the Lagrangian wave interaction equations for adiabatic gas dynamics derived in Webb Brio and Zank (1998), to include the effect of cosmic rays. The resulting wave interaction equations can be shown to be equivalent to the Eulerian wave mixing equations (5). To proceed with the analysis, first note the perturbed mass continuity equation:

$$\frac{\partial\delta\rho}{\partial t} + \frac{\partial}{\partial x}(\rho\delta u + u\delta\rho) = 0, \tag{9}$$

is satisfied automatically by setting

$$\delta\rho = -\frac{\partial}{\partial x}(\rho\xi), \quad \rho\delta u + u\delta\rho = \frac{\partial}{\partial t}(\rho\xi), \tag{10}$$

where ξ is the Lagrangian fluid displacement. From (10), we obtain the formula

$$\delta u = \xi_t + u\xi_x - \xi u_x, \tag{11}$$

for the Eulerian fluid velocity perturbation δu. The Lagrangian perturbation $\Delta\psi$ of a physical quantity ψ and the corresponding Eulerian perturbation $\delta\psi$ are related by

$$\Delta\psi = \delta\psi + \xi\cdot\nabla\psi, \tag{12}$$

(e.g. Newcomb, 1962). Using (10)-(12), we obtain the standard formulae:

$$\Delta\rho = -\rho\frac{\partial\xi}{\partial x}, \quad \Delta u = \xi_t + u\xi_x \equiv \frac{d\xi}{dt}, \tag{13}$$

for the Lagrangian density and fluid velocity perturbations.

From (2), the linearized momentum equation may be written as:

$$\frac{\partial}{\partial t}(\rho\delta u + u\delta\rho) + \frac{\partial}{\partial x}\left(u^2\delta\rho + 2\rho u\delta u + \delta p_g + \delta p_c\right) = 0. \tag{14}$$

In (14), $\delta\rho$ and δu are given by (10) and (11). The gas pressure perturbation δp_g:

$$\delta p_g = a_g^2\delta\rho + p_g\delta\bar{S} \equiv a_g^2\delta\rho + p_g(\Delta\bar{S} - \xi\partial\bar{S}/\partial x), \tag{15}$$

can be written in terms of ξ and $\Delta\bar{S}$, where $\bar{S} = S/C_v$.

By using a multiple scales analysis of the cosmic ray energy equation (4) (see e.g. Webb et al. 1997), we find:

$$\frac{\partial\delta p_c}{\partial x} = \frac{a_c^2}{\kappa}\rho\delta u - \frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\delta\rho, \tag{16}$$

for the cosmic ray pressure gradient perturbation term in (14). In (16), $\partial(\delta p_c)/\partial x \equiv \partial(p_c^{(2)})/\partial x_1$, where $\delta p_c = \epsilon^2 p_c^{(2)}$, $\delta u = \epsilon u^{(1)}$, $\delta\rho = \epsilon\rho^{(1)}$; $x_1 = x/\epsilon$ is the short scale space variable and $\epsilon = \lambda/L$ is the ratio of the short scale λ, characteristic of the wave to the scale length L of the background flow. Note that the perturbations depend on both the slow variables $\bar{x} = x/L$ and $\bar{t} = t/T$, characteristic of the background flow and also on the fast variables $x_1 = \bar{x}/\epsilon$ and $t_1 = \bar{t}/\epsilon$ characteristic of the waves.

Using (10), (11), (15) and (16) in the linearized momentum equation (14) yields the wave equation:

$$\xi_{tt} + 2u\xi_{xt} + (u^2 - a_g^2)\xi_{xx} - \frac{\gamma_g + 1}{\rho}\frac{\partial p_g}{\partial x}\frac{\partial\xi}{\partial x} + Q_{\Delta S} + Q_{cr} = 0, \tag{17}$$

where

$$Q_{\Delta S} = \frac{1}{\rho}\frac{\partial}{\partial x}(p_g\Delta\bar{S}), \tag{18}$$

$$Q_{cr} = \frac{\xi}{\rho\kappa}\left[\frac{\partial}{\partial x}\left(\kappa\frac{\partial p_c}{\partial x}\right) - \gamma_c p_c\frac{\partial u}{\partial x}\right] + \frac{a_c^2}{\kappa}\frac{d\xi}{dt} + \frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\frac{\partial\xi}{\partial x}, \tag{19}$$

are effective source terms due to the Lagrangian entropy perturbations ($Q_{\Delta S}$), and the cosmic ray squeezing instability (Q_{cr}). The perturbed entropy conservation equation (3), yields the advection equation:

$$\frac{\partial\Delta\bar{S}}{\partial t} + u\frac{\partial\Delta\bar{S}}{\partial x} = 0, \tag{20}$$

for the Lagrangian entropy perturbation $\Delta\bar{S}$. By using (4), we obtain the more suggestive form:

$$Q_{cr} = \frac{\xi^{1-\gamma_c}}{\rho\kappa}\frac{d}{dt}\left(p_c\xi^{\gamma_c}\right) + \frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\frac{\partial\xi}{\partial x}. \tag{21}$$

for the cosmic ray squeezing instability term Q_{cr}. The first term on the righthand side of (21) suggests that it is the non-adiabatic squeezing of the fluid element by the cosmic rays $(d/dt(p_c\xi_c^{\gamma}) \neq 0)$ that is primarily responsible for the squeezing instability (note $\zeta = 0$ if $\kappa = const.$).

4.1 Arbitrary Wavelength Waves.

In the more general case of arbitrary wavelength waves, $\xi(x,t)$ again satisfies the the wave equation (17), except that the cosmic ray source term Q_{cr} has the form:

$$Q_{cr} = \frac{1}{\rho}\left(\xi\frac{\partial^2 p_c}{\partial x^2} + \frac{\partial\delta p_c}{\partial x}\right), \tag{22}$$

where δp_c satisfies the perturbed cosmic ray energy equation (4):

$$\begin{aligned}\mathcal{L}(\delta p_c) &\equiv \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \gamma_c\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\kappa(\rho)\frac{\partial}{\partial x}\cdot\right)\right]\delta p_c \\ &= -\left(\delta u\frac{\partial p_c}{\partial x} + \gamma_c p_c\frac{\partial\delta u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\kappa'(\rho)\frac{\partial p_c}{\partial x}\delta\rho\right),\end{aligned} \tag{23}$$

and ΔS satisfies the entropy advection equation (20). Equations (17), (18), (20), (22) and (23) can be combined to yield a single fourth order wave equation for $\xi(x,t)$, which results from simplifying the equation:

$$\mathcal{L}\frac{d}{dt}\left(\xi_{tt} + 2u\xi_{xt} + (u^2 - a_g^2)\xi_{xx} - \frac{\gamma_g+1}{\rho}\frac{\partial p_g}{\partial x}\frac{\partial\xi}{\partial x} + Q_{\Delta S} + Q_{cr}\right) = 0, \tag{24}$$

where $d/dt = \partial/\partial t + u\partial/\partial x$ is the Lagrangian time derivative.

5 Concluding Remarks

The Eulerian density perturbations (7) can be expressed in terms of ξ and ΔS. One can show that the Eulerian wave mixing equations (5) are satisfied if ξ and $\Delta\bar{S}$ satisfy the Lagrangian equations (17)-(20). It is also possible to derive more complicated wave interaction equations for ξ and $\Delta\bar{S}$ for arbitrary wavelength waves (see (22)-(24)).

Acknowledgements. The work of GMW was supported in part by NASA grant NAG5-5164. GPZ is supported in part at the BRI by NSF-DOE Award ATM-9713223, and in part by an NSF Young investigator award ATM 9357861.

References

Drury, L. O'C. & Falle, S.A.E.G. 1986, MNRAS, 223, 353
Newcomb, W. A. 1962, Nuclear Fusion Suppl. (2), 451
Webb, G. M., Brio, M., Zank, G. P. & Story, T. 1997, J. Plasma Phys., 57, 631
Webb, G. M., Brio, M. & Zank, G. P. 1998 J. Plasma Phys., 60, 341
Webb, G. M., Zakharian, A. R. & Zank, G. P. 1999, J. Plasma Phys. (in press)