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The transport of energetic solar particles

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Abstract

A new approach to solving the BGK Boltzmann equation quasi-numerically is described. Our approach shows that low-order expansions can be used to investigate particle propagation at arbitrarily small times in a scattering medium. The characteristic theory of linear hyperbolic partial differential equations illuminates the role of causality. The approach is not restricted to isotropic initial data.

1 Introduction:

The propagation of charged particles along the interplanetary magnetic field (IMF) is governed by the Fokker-Planck equation, where the scattering operator is given either by a quasi-linear pitch-angle model or by a relaxation time approximation. Both forms of the Fokker-Planck equation can be reduced to a diffusion equation describing particle propagation [Jokipii, 1966], which is invalid for short times (with respect to the scattering timescale) after the impulsive release of particles from a source. The infinite signal speed difficulty can be avoided by using instead a telegrapher equation description for the omni-directional phase-space density [Fisk and Axford, 1969; Earl, 1974]. The telegrapher equation, while satisfying causality, cannot however describe the early phases of particle propagation when particles have experienced little or no scattering. In an attempt to circumvent some of these difficulties, Gombosi et al. [1993] expanded the BGK Boltzmann equation asymptotically in a "diffusion parameter" to obtain a higher order modified-telegrapher equation with a phase speed different from the familiar $\pm v/\sqrt{3}$ of the standard telegrapher equation [Fisk and Axford, 1969]. Nonetheless, the modified-telegrapher equation remains invalid at short times and, as we argue below, the modified phase speed does not capture the essential scattered particle propagation characteristics any better than the original telegrapher equation.

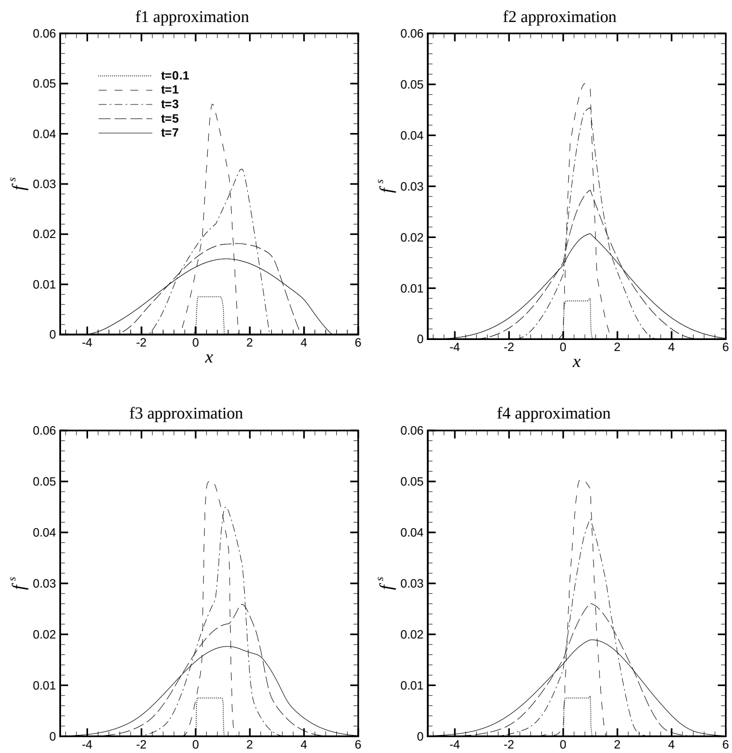

Figure 1: Evolution of the scattered particle distribution f^s for the f_1, f_2, f_3, and f_4 truncations. The full scattered distribution is plotted (i.e., not just f_0). Here $\mu_0 = 0.25$, $v = 1$ (normalized value). The time, normalized to τ, ranges from $t/\tau = 0.1 - 7$.

Important papers by Federov and Shakov [1993] and Kóta [1994] presented exact solutions to the BGK Boltzmann equation which are valid for all times and use simple initial data. This work, while beautiful, does not lend itself readily to the investigation of more complicated initial data or more complicated forms of the Boltzmann equation. Standard numerical approaches to the solution of of the Boltzmann equation do not always provide a satisfactory understanding of the physics underlying particle transport when compared to either the telegrapher or diffusion equation, and are often computationally challenging. Here, we sketch a new approach to solving the BGK Boltzmann equation quasi-numerically, drawing on the rich heritage of charged particle transport in a scattering medium. Examples are presented for one initial condition. Our approach shows that low-order expansions can be used to investigate particle propagation at arbitrarily small times in a scattering medium, and the characteristic theory of linear hyperbolic partial differential equations illuminates the role of causality. Furthermore, unlike existing polynomial expansion methods, arbitrarily anisotropic initial data can be prescribed since we are not restricted to isotropic initial data.

2 Results

In this paper, we consider first particle transport with isotropic scattering. We defer to a separate paper our more general results which include anisotropic scattering, focussing, and convection [Zank et al., 1999]. The BGK Boltzmann equation for isotropic scattering is

$$\frac{\partial f}{\partial t} + \mu v \frac{\partial f}{\partial x} = \frac{f_0 - f}{\tau} + S, \tag{1}$$

where $f = f(x, t, \mu, v)$ is the velocity space distribution function at position x at time t, particle velocity v and pitch-angle cosine $\mu \equiv \cos\theta$; $f_0 \equiv (1/2)\int_{-1}^{1} f \mathrm{d}\mu$ is the isotropic distribution function averaged over μ, $\tau \equiv \tau(x, \mu, v)$ is the collision time, and S is a source term.

The distribution function f can be separated into those particles, denoted by F, which have not experienced scattering, and those which have, denoted by f^s, i.e., $f = F + f^s$. Equation (1) may therefore be expressed as

$$\frac{\partial F}{\partial t} + \mu v \frac{\partial F}{\partial x} = -\frac{F}{\tau} + S, \tag{2}$$

$$\frac{\partial f^s}{\partial t} + \mu v \frac{\partial f^s}{\partial x} + \frac{f^s}{\tau} = \frac{f_0}{\tau} + \frac{F_0}{\tau}, \tag{3}$$

where $F_0 \equiv (1/2)\int_{-1}^{1} F \mathrm{d}\mu$. Obviously, F experiences losses only and has the exact solution

$$F(x, t, \mu, v) = F(x - \mu v t, 0, \mu, v) e^{-t/\tau}. \tag{4}$$

To solve (3), we may expand f^s in an infinite series of Legendre polynomials $P_n(\mu)$,

$$f^s = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) P_n(\mu) f_n, \tag{5}$$

where f_n is the nth harmonic of the scattered distribution function.

The f_1 approximation (i.e., assume $f_n = 0 \; \forall \, n \geq 2$) yields the inhomogeneous telegrapher equation

$$\tau \frac{\partial^2 f_0}{\partial t^2} + \frac{\partial f_0}{\partial t} - \kappa \frac{\partial^2 f_0}{\partial x^2} = 2\pi F_0^*, \qquad \kappa \equiv \frac{v^2 \tau}{3}, \tag{6}$$

with the usual signal propagation speed $\pm v/\sqrt{3}$, and $F_0^* = (1/2)\int_{-1}^{1} \partial F/\partial t \mathrm{d}\mu$. The inhomogeneous term in (6) is the source term for the isotropic scattered component and the initial data is only with respect to F in equation (4) and can be arbitrarily anisotropic. Equation (6) can be solved analytically and the solution grows up gradually from $f_0 = 0$ at the initial time. Unlike the standard telegrapher equation, no coherent propagating pulses are present in the solution of (6), and the solution is valid for arbitrarily small times.

The expressing of the truncated infinite system of expanded equations derived from (4) by (6) is not especially revealing. The expanded system of equations in the f_n approximation forms a linear hyperbolic system with an infinite (discrete) spectrum of characteristic speeds, expressed as

$$\Psi_t + vA\Psi_x = C, \tag{7}$$

where $\Psi = (f_0, f_1, f_2, \dots f_n)^t$ and A is the tridiagonal matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & & & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 & \dots & & & 0 \\ 0 & 2/5 & 0 & 3/5 & 0 & \dots & & & 0 \\ 0 & 0 & 3/7 & 0 & 4/7 & \dots & & & 0 \\ \vdots & & & & & \vdots & & & \vdots \\ \dots & & & & & \dots & \frac{n-1}{2n-1} & 0 & \frac{n}{2n-1} \\ \dots & & & & \dots & 0 & \frac{n}{2n+1} & & 0 \end{pmatrix}, \tag{8}$$

$$\tau C = (2\pi F_0^*, f_1, f_2, \dots, f_n)^t.$$

The $(n+1)$ characteristics of (7) are all distinct. When n is even, the number of characteristics is odd and consists of $n/2$ propagating information forward, $n/2$ propagating information backward, and one that is stationary. For example, the f_2 characteristics are $(\mathrm{d}x/\mathrm{d}t)_{0,\pm} = 0, \pm\sqrt{3/5}v$. Whereas at the f_1 (telegrapher) level, all scattered particles propagate at $\pm v/\sqrt{3}$, the f_2 approximation is more refined, substituting $0, \pm\sqrt{3/5}v$ for the speeds of the scattered particles. Since only the even truncations f_n (n even) admit the zero characteristic, these are the most accurate solutions to (1). We find further that the f_2 approximation is an entirely adequate truncation.

Solutions to (1) for an initial beam distribution $\propto \delta(\mu - \mu_0)$ for the f_1, f_2, f_3, and f_4 approximations are illustrated in Figure 1. These solutions were obtained using a numerical method of characteristics scheme. The figures show only the scattered part of the distribution function f^s from an early time ($t/\tau = 0.1$) until a later time ($t/\tau = 7$). The initial beam had $\mu_0 = 0.25$ and was localized between $x = 0$ and 1. At early times, a flat box-like distribution of scattered particles is generated as the initial propagating beam decays. The distribution peaks at slightly later times ($t/\tau = 1$) while remaining very localized. The scattered particle distribution continues to grow and spread out in time. The differences between the odd and even truncations are apparent. The f_2 approximation proves to be a good truncation to the full solution and few differences exist between the f_2 and f_4 solutions. Finally, none of the solutions, including the f_1 telegrapher truncation, of Figure 1 exhibit oppositely directed propagating pulses.

The above approach is extended easily to anisotropic scattering. We assume that scattering through 90° is slow and, following Kóta [1994], introduce two scattering timescales τ_1 and τ_2. In the $\mu < 0$ and $\mu > 0$ hemispheres, particle scattering is isotropic occurring at a rate τ_1^{-1}. Particle scattering from one hemisphere to another proceeds at the slower rate τ_2^{-1}. Let $f^\pm$ denote forward/backward moving particles, allowing one to generalize (1) to [Kóta, 1994]

$$\frac{\partial f^-}{\partial t} + \mu v \frac{\partial f^-}{\partial x} = \frac{f_0^- - f^-}{\tau_1} + \frac{f_0^+ - f^-}{\tau_2}; \qquad \frac{\partial f^+}{\partial t} + \mu v \frac{\partial f^+}{\partial x} = \frac{f_0^+ - f^+}{\tau_1} + \frac{f_0^- - f^+}{\tau_2}, \tag{9}$$

where $f_0^- \equiv (1/2)\int_{-1}^{0} f\,\mathrm{d}\mu$, $f_0^+ \equiv (1/2)\int_0^{-1} f\,\mathrm{d}\mu$. We again separate $f^\pm$ into scattered and unscattered populations to obtain two coupled sets of equations analogous to (2) and (3). As before, the unscattered distribution can be solved exactly and it decays with time. A half-range expansion for the scattered particles can be developed and the resulting linear hyperbolic system can be solved using characteristics. The $f_1^{\pm}$ truncation, for example, admits four signal speeds $\lambda_{\pm}^{+} = \left(1 \pm 1/\sqrt{3}\right) v/2$, $\lambda_{\pm}^{-} = -\left(1 \pm 1/\sqrt{3}\right) v/2$.

For two initial ring beam distributions propagating in opposite directions ($\mu_0 = \pm 0.25$) and $\tau_2/\tau_1 = 10$ (i.e., highly anisotropic scattering), we show in Figure 2 the evolution of the scattered particle distribution using an $f_2^{\pm}$ approximation. Initially, persistent oppositely propagating coherent pulses exist which later merge to form a broad distribution. The solutions presented here are identical to the exact analytic solutions of Kóta [1994] (his Figure 4).

We may conclude by noting that a new and relatively simple method for solving the BGK Boltzmann equation has been presented. A virtue of this approach is its close connection to existing diffusion and telegrapher equation models.

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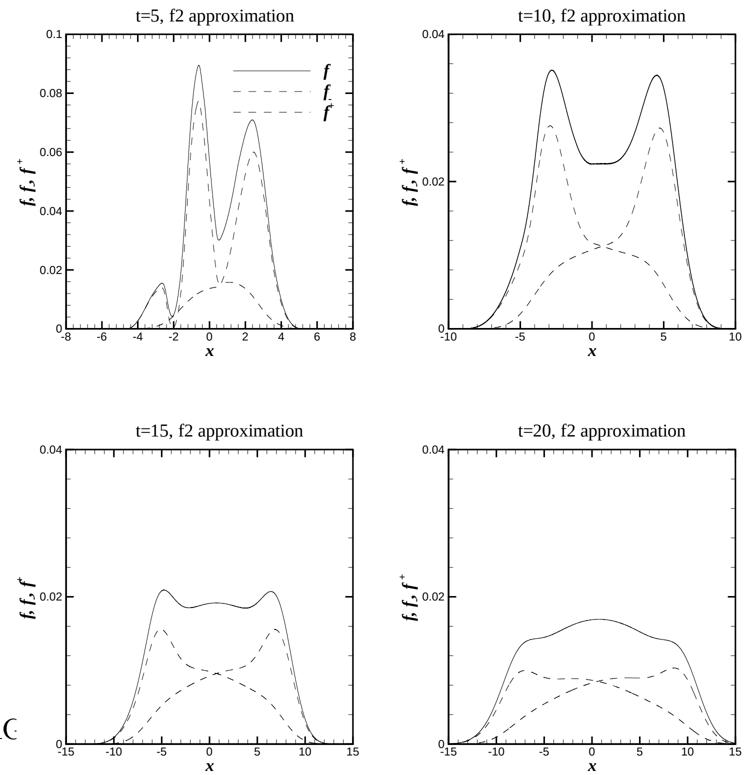

Figure 2: Evolution of the scattered particle distribution $f^{\pm}$ for the f_2 truncation. The full scattered distribution f^s is plotted together with f^+ and f^-. Here $\mu_0 = \pm 0.25$, $v = 1$ (normalized value), and $\tau_2/\tau_1 = 10$ for a range of times.

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