Multiple Scattering and the BGK Boltzmann Equation

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Abstract

A multiple scattering formalism is developed for the BGK Boltzmann equation. The formalism is equivalent to constructing the Neumann series for the corresponding integral equation. By using transform methods, the distribution of unscattered particles f_0, and the distribution of particles that have undergone n-scatters, f_n ($n \geq 1$), are determined for the initial value problem in a uniform background medium. The method is used to investigate solutions of the BGK Boltzmann equation considered by Fedorov and Shakov (1993) and Kota (1994) in studies of coherent and diffusive particle transport.

1 Introduction

The BGK Boltzmann equation has been used to elucidate various aspects of cosmic ray transport in astrophysical settings. These include the derivation of diffusive transport equations for cosmic rays (e.g. Earl, Jokipii and Morfill, 1988) that generalize the Parker transport equation to include cosmic ray viscosity and accelerating reference frame effects; models of coherent and non-diffusive particle transport (Fedorov and Shakov, 1993; Kota, 1994); and nonlinear, one fluid models of cosmic ray modified shocks (Berezhko et al. 1983). The main purpose of this paper is to investigate coherent and non-diffusive particle transport (e.g. Fedorov and Shakov, 1993; Kota, 1994) using a multiple scattering formalism.

2 Model and Equations

Our main interest in this paper is with solutions of the BGK Boltzmann equation:

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial x} = \frac{\langle f \rangle - f}{\tau} + Q, \tag{1}$$

where

$$\langle f \rangle = \tfrac{1}{2} \int_{-1}^{1} f(x, t; v, \mu) d\mu, \quad \mu = \cos\theta, \tag{2}$$

subject to the initial condition:

$$f(x, t; v, \mu) = A(x) B(\mu) \quad \text{at time t} = 0. \tag{3}$$

In the following analysis, the particle speed v is taken to be a constant parameter. In the above equations, $f(x, t; v, \mu)$ is the velocity space distribution function at position x and time t, for particles with speed v and pitch angle cosine $\mu = \cos\theta$; $\langle f \rangle$ is the mean distribution function averaged over μ, and $\tau = \tau(x, v)$ is the collision time. The choices $A = N\delta(x)$ and $B(\mu) = \delta(\mu - \mu_0)$, correspond to the solution of Fedorov and Shakov (1993). We consider the case $Q \equiv 0$, and take $\tau = \tau(v)$ to be independent of x.

3 Integral Equations and Multiple Scattering Solutions

Formally integrating the characteristics for (1) with $Q = 0$, results in the integral equation:

$$f(x, t; v, \mu) = \int_0^t \langle f \rangle (x', t', v) \exp[-(t - t')/\tau] \, dt' + f(x - v\mu t, 0; v, \mu) \exp(-t/\tau), \tag{4}$$

where

$$x' = x - v\mu(t - t'), \tag{5}$$

denotes the position of the particle at the last scatter. The second term on the righthand side of (4) represents the unscattered particles and $P(t) = \exp(-t/\tau)$ is the probability that the initial particles have not been scattered at time t. Equation (4) is an integral equation for f in which $\langle f\rangle$ is given by (2). One can also average (4) over μ to obtain the integral equation:

$$\langle f\rangle(x,t,v) = \tfrac{1}{2}\int_{-1}^{1} d\mu \left(\int_0^t \langle f\rangle(x',t',v)\exp[-(t-t')/\tau]\,dt' + f(x-v\mu t, 0; v,\mu)\exp(-t/\tau)\right) \qquad (6)$$

for $\langle f\rangle$ (c.f. also Berezhko et al. 1983).

To develop a multiple scattering formalism for the BGK Boltzmann equation (1), with $Q=0$, i.e.,

$$\frac{df}{dt} = \frac{\langle f\rangle - f}{\tau}, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + v\mu\frac{\partial}{\partial x}, \qquad (7)$$

we write the solution for f in the form:

$$f = f_0 + f_c, \quad f_c = \sum_{n=1}^{\infty} f_n, \qquad (8)$$

where f_0 denotes the distribution of unscattered particles and f_n is the distribution of particles that have undergone n-scatters (e.g. Kuhn, 1979). From physical reasoning, the partial distribution functions $\{f_j : j = 0,1,2,\ldots\}$ satisfy the coupled evolution equations:

$$\frac{df_0}{dt} = -\frac{f_0}{\tau}, \quad \frac{df_n}{dt} = \frac{\langle f_{n-1}\rangle}{\tau} - \frac{f_n}{\tau}, \quad n \geq 1. \qquad (9)$$

In (9), the particles that have undergone $(n-1)$ scatters provide a source for the particles that will undergo n-scatters. The initial value data for (9) are:

$$f_0 = A(x)B(\mu), \quad f_n = 0, \quad n \geq 1, \quad \text{at time t} = 0, \qquad (10)$$

since there are no scattered particles at time $t=0$. The Neumann series of the Volterra type integral equation (4) for f is obtained by using the iteration scheme:

$$f^{(n)} = K[f^{(n-1)}] + f_0, \quad f^{(0)} = f_0, \qquad (11)$$

where $f^{(n)}$ is the n^{th} iterate and K denotes the integral (and averaging) operator in (4). From (11),

$$f^{(N)} = (I + K + K^2 + \ldots + K^N)f_0 = f_0 + \sum_{n=1}^{N} f_n, \qquad (12)$$

where $f_n = K^n(f_0)$ is the distribution of particles that have undergone n-scatters. Letting $N\to\infty$ we obtain (8).

From (4), or by using transform methods we obtain

$$f_0(x,t,v,\mu) = \exp(-\bar{t})f_0(x-v\mu t,0;v,\mu) = \exp(-\bar{t})A(x-v\mu t)B(\mu), \qquad (13)$$

for the distribution of unscattered particles, where in (13) and below we use the dimensionless variables $\bar{t} = t/\tau$ and $\bar{x} = x/(v\tau)$. Using transform methods yields the solution for f_n in the form:

$$f_n = \int_{-1}^{1} d\mu_0 B(\mu_0)\int_{-\infty}^{\infty} dx_0\ A(x_0)G_n(\bar{x},\bar{t},v,\mu;\bar{x}_0,\mu_0), \quad (n\geq 1), \qquad (14)$$

where the Green's function $G_n(\bar{x},\bar{t},v,\mu;\bar{x}_0,\mu_0)$ is given by

$$
\begin{aligned}
G_n(\bar{x},\bar{t},v,\mu;\bar{x}_0,\mu_0) \quad = \quad & \frac{\exp(-\bar{t})}{2v\tau(\mu-\mu_0)}\Bigg[-\frac{1}{\pi}PV\Bigg\{\int_{-1}^{1}\left(\frac{1}{p-\mu}-\frac{1}{p-\mu_0}\right)V_n^-(\bar{x},\bar{t},\bar{x}_0,p) \\
& [H(\bar{x}-\bar{x}_0H(p)-H(\bar{x}_0-\bar{x})H(-p)]\,dp\Bigg\} \\
& +[V_n^+(\bar{x},\bar{t},\bar{x}_0,\mu)[H(\bar{x}-\bar{x}_0)H(\mu)-H(\bar{x}_0-\bar{x})H(-\mu)] \\
& -V_n^+(\bar{x},\bar{t},\bar{x}_0,\mu_0)[H(\bar{x}-\bar{x}_0)H(\mu_0)-H(\bar{x}_0-\bar{x})H(-\mu_0)]\Bigg], \qquad (15)
\end{aligned}
$$

($n \geq 1$). The functions $V_n^{\pm}(\bar{x},\bar{t},\bar{x}_0,p)$ in (15) are defined by the equations:

$$
V_n^-(\bar{x},\bar{t},\bar{x}_0,p) = \Im\left([m(p)+i\pi]^{n-1}\right)\frac{[\frac{1}{2}(\bar{x}-\bar{x}_0-p\bar{t})]^{n-1}}{(n-1)!}H\left(\bar{t}-\frac{\bar{x}-\bar{x}_0}{p}\right), \qquad (16)
$$

$$
V_n^+(\bar{x},\bar{t},\bar{x}_0,p) = \Re\left([m(p)+i\pi]^{n-1}\right)\frac{[\frac{1}{2}(\bar{x}-\bar{x}_0-p\bar{t})]^{n-1}}{(n-1)!}H\left(\bar{t}-\frac{\bar{x}-\bar{x}_0}{p}\right), \qquad (17)
$$

where $m(p) = \ln|(1-p)/(1+p)|$. In the above equations $H(x)$ denotes the Heaviside step function. The symbol PV in front of the integral in (15) denotes a Cauchy principal value (CPV) integral with possible singularities at $p=\mu$ and $p=\mu_0$, and $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of the complex number z. Note that the singularities at $p=\pm 1$, are integrable. The functions $V_n^{\pm}(\bar{x},\bar{t},\bar{x}_0,p)$ are only non-zero for $|p|\bar{t} > |\bar{x}-\bar{x}_0|$, which corresponds to the causality constraint and implies $|(\bar{x}-\bar{x}_0)/\bar{t}| < |p| < 1$ in the CPV integral.

The total scattered distribution f_c is given by:

$$
f_c = \sum_{n=1}^{\infty} f_n = \int_{-1}^{1} d\mu_0\; B(\mu_0)\int_{-\infty}^{\infty} dx_0\; A(x_0)G_c(\bar{x},\bar{t},v,\mu;\bar{x}_0,\mu_0), \qquad (18)
$$

where $G_c = \sum_{n=1}^{\infty} G_n$ is the Green's function for the total scattered distribution. One can show that G_c is equivalent to the Green's function for the scattered particles given by Fedorov and Shakov (1993), which consists of a superposition of slowly decaying diffusive eigenmodes, which dominate at late times, plus a combination of fast decaying eigenmodes, which decay exponentially in time as $\exp(-\bar{t})$ (see also Kota (1994)). The slowly decaying diffusive eigenmodes may be identified in part with the CPV integral contributions to G_n in (15). At late times the solution for G_c is approximately given by the Green's function for the diffusion equation, i.e. $G_c \simeq \exp[-(x-x_0)^2/(4\kappa t)]/(4\pi\kappa t)^{\frac{1}{2}}$, where $\kappa = \frac{1}{3}v^2\tau$ is the diffusion coefficient. At early times the scattered distribution is dominated by the particles that have undergone a single scatter (i.e., f_1 or G_1).

4 Numerical examples and discussion

Figure 1 illustrates the partial sum $G_c^{(N)} = \sum_{n=1}^{N} G_n$, of particles that have done at most N-scatters (the dash-dot curves), and the distributions $\{G_n\}$, as functions of $\bar{t}$. The parameters are: $\bar{x}_0 = 0$, $\bar{x} = 1$, $\mu_0 = 0.99$, and the distributions are shown for $\mu = 0$, $\mu = 0.49$ and $\mu = 0.98$ (panels a, b and c). For $\mu = 0$ and $\mu = 0.49$, the figures show the sum of $N = 13$ terms, but $N = 8$ in the $\mu = 0.98$ case. Panel (d) shows again the partial sums $G_c^{(N)}$ for $\mu = 0, 0.49$ and 0.98; the bold dots show for comparison the Fedorov-Shakov solution results at three separate times. The $G_c^{(N)}$ exhibit diffusive behaviour ($G_c \propto \bar{t}^{-\frac{1}{2}}$) at large $\bar{t}$, whereas G_1 shows coherent propagation characteristics ($G_1 \neq 0$ for $\bar{x} < \bar{t} <$ max$(\bar{x}/|\mu|, \bar{x}/|\mu_0|)$; note also there is no CPV contribution to G_1). The calculations show that the distributions are highly anisotropic at early times, but become almost isotropic at late times. There were some numerical accuracy problems in evaluating the CPV integrals for the $\mu = 0.98$ case for $\bar{t} > 4.5$ in panel (d), which hopefully can be improved upon. The formula (15) does not apply for $\mu = 1$, or for $\mu = \mu_0$; further work is needed to clarify these singular cases. To deal with the $p = \pm 1$ singularities in (15) it is useful to use the integration variable $y = -\ln(1 - |p|)$, and a special strategy is used to isolate the CPV singularities.

Acknowledgements. We acknowledge stimulating discussions with Jozsef Kota. GMW was supported in part by NASA grant NAG5-5164. GPZ is suppported in part by NSF-DOE Award ATM-9713223, and by an NSF Young investigator award ATM 9357861.

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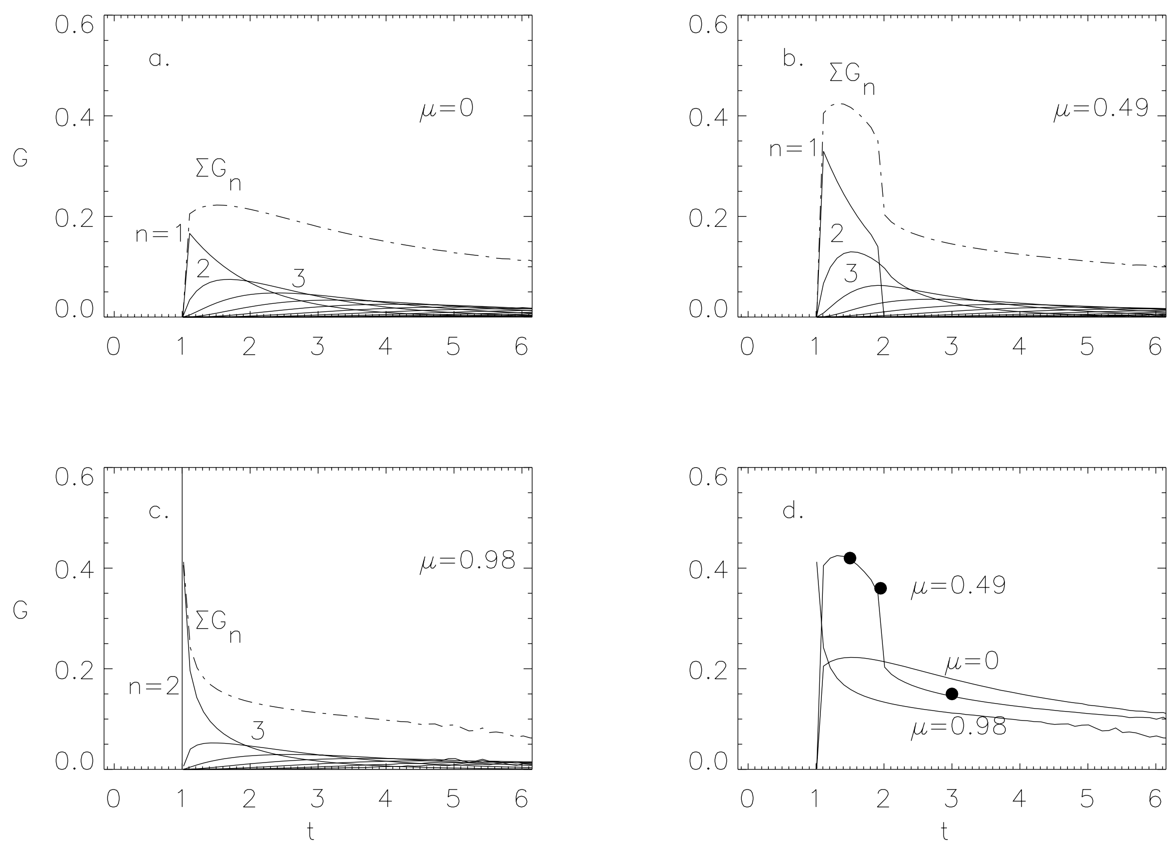

Figure 1: Partial sums $\sum_{n=1}^{N} G_n$, and distributions G_n versus $\bar{t}$.