Dispersion Relations for Diffusive Motion

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Abstract

We discuss the dispersion relations for diffusive motion. These can be quickly computed and can be a useful diagnostic tool to explore the validity range of various approximations. Illustrative examples are presented for cases including dominant helicity, focusing, and hemispherical scattering.

1 Introduction:

The evolution of the distribution, $f(z, \mu, t)$ for impulsive particle events in time, t, space, z, and cosine of pitch-angle, μ is governed, in the simplest rectilinear geometry, by the equation

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} D_{\mu\mu} \frac{\partial f}{\partial \mu} \tag{1}$$

where $D_{\mu\mu}$ is the pitch-angle scattering coefficient. (1) is often approximated by the diffusion equation which operates with the omnidirectional density $f_0(z, t) =< f(z, \mu, t) >$ only ($<>$ indicates average over μ). The diffusion model is inaccurate for short times and fails to describe the early phase of events. Several efforts have been made to improve the diffusion model. Fisk & Axford (1969) introduced the telegrapher's equation. The modified equations can be written in the general form of

$$\alpha \frac{\partial^2 f_0}{\partial t^2} + \Lambda \frac{\partial^2 f_0}{\partial z \partial t} + \frac{1}{\tau} \frac{\partial f_0}{\partial t} = \frac{1}{A} \frac{\partial}{\partial z} \Big(\frac{A v^2}{3} \frac{\partial f_0}{\partial z} \Big) = \frac{v^2}{3} \Big(\frac{\partial^2 f_0}{\partial z^2} + \frac{1}{L} \frac{\partial f_0}{\partial z} \Big) \tag{2}$$

where $\alpha = 0, \Lambda = 0$ corresponds to the standard diffusion equation, $\alpha = 1, \Lambda = 0$ yields the telegrapher's equation. Pauls et al (1993) suggested a cross-derivative term ($\Lambda \neq 0$) to account for a possible dominant helicity in the random component of the heliospheric magnetic field. Gombosi et al (1993) pointed out that a modified α, which depends on the actual form of $D_{\mu\mu}$ gives better approximation. The right hand side of (2) includes adiabatic focusing due to the possible divergence of field lines, A(z) is the area element, and L is the focusing length, with $1/L = \partial lnA/\partial z$).

One way to explore these approximations is to look at the dispersion relations they yield. Consider the solution, $f(z, \mu, t)$ as a sum of eigenfunctions, $F(k, \mu)e^{(ikz - \nu t)}$. Then, (2) transforms into

$$\alpha \nu^2 + i\Lambda k \nu - \nu/\tau = -v^2(k^2 - ik/L)/3 \tag{3}$$

The resulting $\nu(k)$ relations are easy to evaluate and are shown in Figure 1. Here we cover only the half plane, obviously $Re(\nu)$ is an even and $Im(\nu)$ is an odd function of k. To use dimensionless quantities, we take the particle speed, v, and the scattering time, τ, to be unity ($v = \tau = 1$). These dispersion relations of the approximations can then be compared to those obtained from the full equation (1). In general, the full equation has infinite number of eigenfunctions and eigenvalues (see Earl, 1974). Here we focus on the two lowest eigenvalues which are the most important in determining the evolution of the particle density and anisotropy. Clearly the value of α appears in $\nu_1(k = 0)$, while a non-zero Λ would appear as a non-zero (imaginary) value of $d\nu_1/dk$ at $k = 0$. The dispersion relations for the 'billiard-ball' scattering were first given by by Fedorov and Shakov (1993). 'Hemispherical' scattering was considered by Kóta (1994).

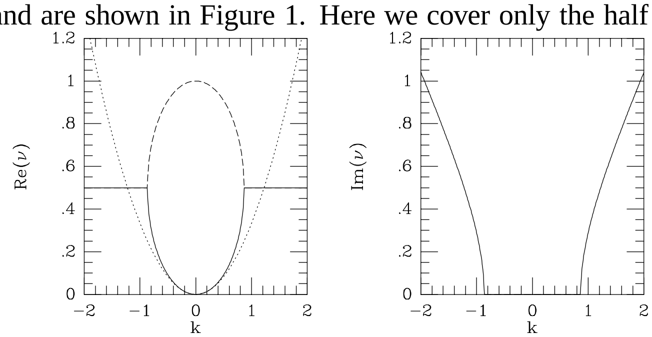

Figure 1: Dispersion relations for the diffusion (dotted line) and the telegrapher's equation (solid lines)

2 Dispersion Relations:

For the sake of simplicity we assume that both $D_{\mu\mu}$ and the focusing length, L, are independent of location, which correspponds to an exponentially diverging geometry (see, Earl 1981). The pitch-angle scattering coefficient, $D_{\mu\mu}$ is allowed to be arbitrary function of μ, we assume

$$D_{\mu\mu} = \frac{v(1-\mu^2)}{2\lambda(\mu)} = \frac{(1-\mu^2)|\mu|^q}{(1\mp\sigma)(1-q)(3-q)\tau} \tag{4}$$

In this formulation, σ accounts for helicity (Bieber et al., 1987) and τ represents the effective scattering time so that the resulting spatial diffusion coefficient be $\kappa_{||} = v^2\tau/3$ (Hasselman & Wibberenz). The Fokker-Planck equation, including focusing, can then be rewritten as

$$\frac{\partial f}{\partial t} + v\mu\frac{\partial f}{\partial z} = -\frac{v(1-\mu^2)}{2L}\frac{\partial f}{\partial \mu} + \frac{\partial}{\partial\mu}D_{\mu\mu}\frac{\partial f}{\partial\mu} = e^{G(\mu)}\frac{\partial}{\partial\mu}\Big(e^{-G(\mu)}D_{\mu\mu}\frac{\partial f}{\partial\mu}\Big) \tag{5}$$

where $\partial G/\partial\mu = \lambda(\mu)/L$ (Kunstmann, 1979). In terms of the eigenfunctions, $F(k,\mu)$, (6) reads as

$$-\nu F + ikv\mu F - e^{G(\mu)}\frac{\partial}{\partial\mu}\Big(e^{-G(\mu)}D_{\mu\mu}\frac{\partial F}{\partial\mu}\Big) = 0 \tag{6}$$

The eigenvalues, $\nu = \nu_j(k)$ ($j = 0,1,2,..$), are complex in general. Slow spatial variation corresponds to $k \approx 0$. At $k = 0$, the lowest eigenvalue is always $\nu_0 = 0$, corresponding to the completely homogeneous and isotropic solution, and all the other eigenvalues are real and the eigenfunctions are identical with the eigenfunctions of the scattering operator (Earl, 1974). Moving from k to $k + \delta k$, the eigenfunctions and eigenvalues change to $F + \delta F$, and $\nu + \delta\nu$, yielding

$$-\nu\delta F + ikv\mu\delta F - e^{G(\mu)}\frac{\partial}{\partial\mu}\Big(e^{-G(\mu)}D_{\mu\mu}\frac{\partial\delta F}{\partial\mu}\Big) = \delta\nu F - i\delta kv\mu F \tag{7}$$

Multiplying equation (7) by $F(k,\mu)$ and integrating over μ, the left hand side should vanish

$$\int_{-1}^{1}(-\delta\nu F^2 + i\delta kv\mu F^2)e^{-G}\,d\mu = 0 \tag{8}$$

hence the variation of $\nu(k)$ is given by

$$\frac{d\nu}{dk} = iv\frac{<\mu e^{-G}F^2>}{<e^{-G}F^2>} \tag{9}$$

The derivative $d\nu/dk$ is the group velocity which can be associated with the coherent propagation speeds while the second derivative $d^2\nu/dk^2$ is characteristic of the dispersion and can be associated with the diffusion coefficients. It can be shown that, for the rectilinear case, $d^2\nu/dk^2$ at $k = 0$ exactly returns the diffusion coefficient derived by Hasselman & Wibberenz (1970).

3 Illustrative Examples:

Below we present some examples to illustrate the method for various kinds of scattering. We assume the dependence of $D_{\mu\mu}$ as given in (5). Clearly, $q = 0, \sigma = 0$ describes isotropic scattering, $\sigma \neq 0$ implies dominant helicity, while $q \approx 1$ represents hemispherical scattering. We consider both rectilinear and focusing geometries, with a constant focusing length, L.

3.1 Isotropic Pitch-angle scattering: The simplest scattering is isotropic pitch-angle scattering. Then the eigenfunctions at $k = 0$ are the spherical harmonics, while the eigenvalues are $\nu_j = j(j+1)/2\tau$ ($j = 0,1,2,...$). The variations of $\nu_0(k)$ and $\nu_1(k)$ as function of k are shown in Figure 2. The general pattern is similar to that of the telegrapher's equation but there are noticeable differences at the same time.

3.2 Dominant Helicity: Bieber et al, Evenson & Matthaeus (1987) called attention to the possible role of a dominant helicity, which introduces an asymmetry into $D_{\mu\mu}$. Figure 3 shows how the dispersion relations for a dominant helicity $\sigma = 0.5$. Note that the imaginary part of the derivative $d\nu_1/dk$ becomes finite at $k = 0$ in accord with the predictions of (2) for a non-zero value of Λ (Pauls et al., 1993).

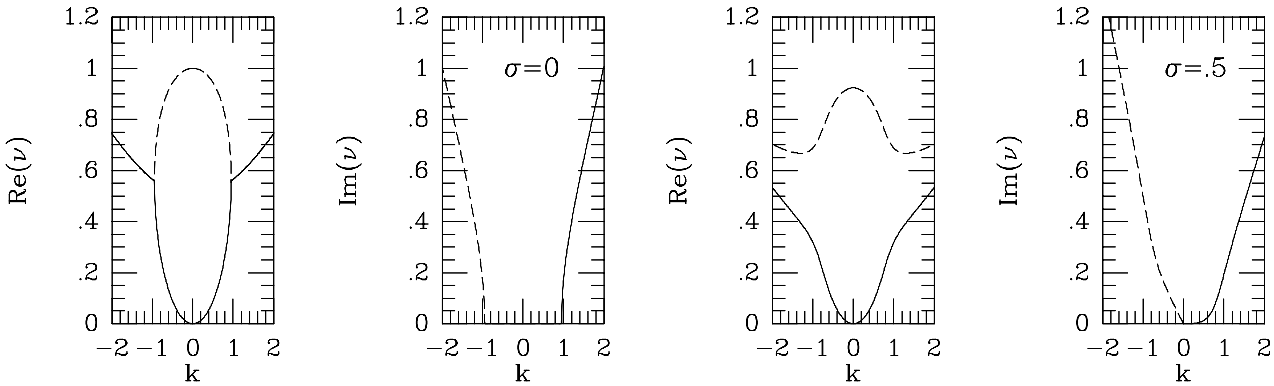

Figure 2: $\nu_0(k)$ (solid line) and $\nu_1(k)$ (dashed line) for isotropic pitch-angle scattering without a dominant helicity ($\sigma = 0$, left) and with a dominant helicity ($\sigma = 0.5$, right)

3.3 Focusing: Adiabatic focusing becomes important when field lines diverge on a scale comparable or smaller than the scattering mean free parth. Focusing appears in (7) through the function, $G(\mu)$

In a focusing geometry, (3) suggests that κ can be obtained as $\kappa = -iL\, d\nu_0/dk$, at $k = 0$. Since the zeroeth eigenfunction, F_0, is always constant at $k = 0$, (9) immediately leads to $\kappa = -vL < \mu e^{-G} > / < e^{-G} >$, which is identical to the expression inferred by Bieber and Burger (1990) using a Born approximation. Bieber, Evenson & Matthaeus (1987) pointed out that the combined effect of focusing and dominant helicity leads to charge dependence in κ. This effect is clearly demonstrated in the dispersion relations shown in Figure 3 for a focusing length, $L = 1$, and helicities, $\sigma = 0.5$ and $\sigma = -0.5$. Both the curvature of $Re(\nu_0)$, and the slopes of $Im(\nu_0)$ at $k = 0$ indicate different effective diffusion coefficients for the two different signs of helicity.

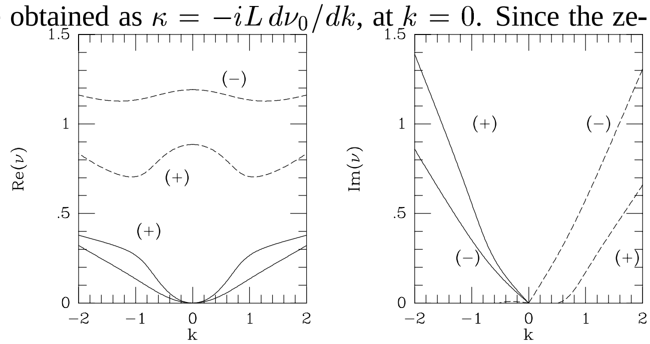

Figure 3: Dispersion relations for a focusing geometry ($L = 1$), with $\sigma = +0.5$ (+) and $\sigma = -0.5$ (-) helicities

4 Hemispherical Scattering:

A case of particular importance is the hemispherical scattering when particles are strongly scattered both in the $\mu < 0$ and $\mu > 0$ hemispheres but scattering through $\mu = 0$ is restricted. Such a case is described, for instance, by $q \approx 1$ or, in another formulation with the introduction of two distinct levels f_+ and f_- for the two hemispheres. The equations for $f_\pm$ have been developed and discussed in detail by Isenberg (1997) and Schwadron (1998).

Figure 5 shows the dispersion relations for $q = 0.9$ for a rectilinear case, without focusing (left panels), and those for a focusing scenario ($L = 1.$). For the rectilinear case, the dispersion relations are quite similar to those of the telegrapher's equation (see Figure 1). Moreover, the higher eigenvalues, ν_j ($j = 2, 3, ..$) are remarkably large. For instance, the second eigenvalue is already $\nu_2 \approx 23$, thus the contributions from the higher eigenfunctions vanish quickly and can be neglected. This also reaffirms that the use of the two distinct levels, f_- and f_+ is a good approximation. For the evolution of f_+ and f_-, we suggest the coupled equations

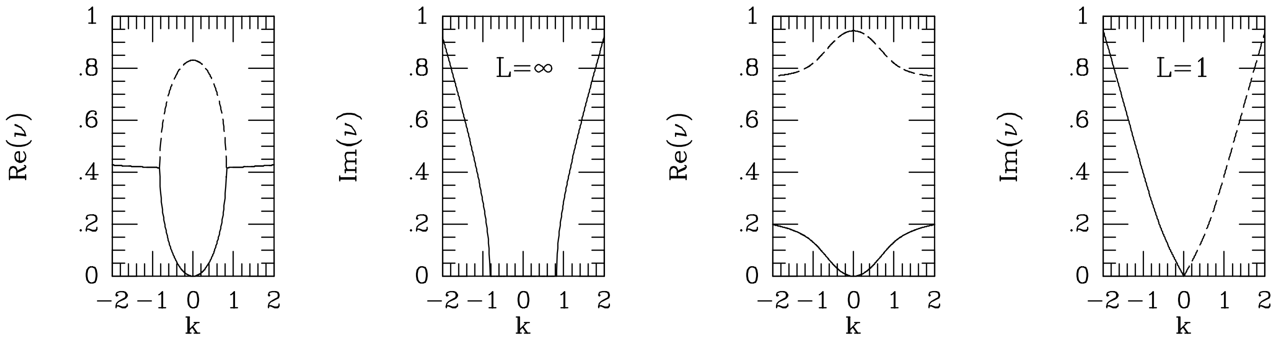

Figure 4: Hemispherical scattering ($q = 0.9$) for rectilinear (left) and focusing ($L = 1$, right) geometries

$$\frac{\partial f_-}{\partial t} - \frac{v}{2}\frac{\partial f_-}{\partial z} = -\frac{f_- - f_+}{T} \tag{10}$$

$$\frac{\partial f_+}{\partial t} + \frac{v}{2}\frac{\partial f_+}{\partial z} + \frac{v}{2L}(f_+ - f_-) = -\frac{f_+ - f_-}{T} \tag{11}$$

which, in the focusing term, are somewhat different from the equations of Schwadron (1998) and Isenberg (1997). Combining (10) and (11) leads to

$$\nu^2 - \left(\frac{v}{2L} + \frac{2}{T}\right)\nu + \frac{v^2}{4}\left(k^2 - \frac{ik}{L}\right) = 0 \tag{12}$$

implying a modification of $\nu_0(k)$ due to the effect of focusing. The right panels of Figure 5 shows that, as expected from (12), the dispersion $d^2\nu/dk^2$ does indeed decrease in the presence of focusing.

5 Summary:

We have presented some illustrative examples how the dispersion relations, which can be computed quickly, can be used to explore some characteristic features of the full transport equation and to diagnose various approximations.

6 Acknowledgement:

The author benefited from discussions with G. Erdős, J.R. Jokipii, and G.M. Webb. This work has been supperted by NASA under Grants NAG5-4834 and NAG5-6620 and by NSF under Grant ATM-9616547.

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