A Path Integral Solution to the Stochastic Differential Equation of the Markov Process for Cosmic Ray Transport

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Abstract

Cosmic ray transport in interplanetary or interstellar magnetic fields can be viewed as a Markov stochastic process and the transport equation has therefore recently been reformulated with a set of stochastic differential equations that describe the guiding center and the momentum of individual charged particles. The Fokker-Planck diffusion equation for the cosmic ray flux can be derived from these stochastic differential equations. Alternatively, the Fokker-Planck equation, like the Schroedinger equation in quantum mechanics, can be solved with a path integral method. Both new methods enable us to solve modulation, propagation and acceleration problems for cosmic ray spectra. In addition, both can reveal insights into the physical processes behind the solutions to these problems since they follow the trajectory and the momentum of individual particles. In this paper, we derive a path integral representation from the stochastic differential equations and thus prove that the two new methods are consistent with each other.

1 Introduction:

Cosmic ray transport in interplanetary or interstellar magnetic fields is often studied in the framework of diffusion models (e.g. Parker, 1965; Ginzburg & Ptuskin, 1975). For interplanetary transport, a Fokker-Planck diffusion equation for the isotropic part of the cosmic ray distribution function can be derived from the collisionless Boltzmann equation with the help of observations of interplanetary magnetic fields (Skilling, 1976). The mechanism for motion of cosmic rays in the interstellar medium is not yet clear simply because of insufficient information on the interstellar medium and the galactic magnetic field. But on the overall scale size of the galaxy and on the time scale of cosmic ray life time ($\sim 10^7$ years), the diffusion approximation seems to be a suitable approach, because it is consistent with the observation of small cosmic ray flux anisotropy and large amount of secondly produced nuclei in cosmic rays relative to the interstellar medium composition. In addition, acceleration of cosmic rays by astrophysical shocks may also be studied in the framework of diffusion models (Drury, 1983).

In this paper, we use stochastic differential equations that describe Markov stochastic processes to replace the diffusion equation as the fundamental transport equation of cosmic rays. From the stochastic differential equation we discretize the stochastic process to get a path integral solution for the transition probability, which is consistent with the Green's function of the diffusion equation. We find a Lagrangian, which, if minimized, describes the most probable trajectory of particles in diffusion process. We prove that the path integral derived from the Markov stochastic process is consistent with the path integral derived from the diffusion equation with quantum mechanics method (Zhang, 1999a). Both the stochastic process method and path integral approach give excellent results for cosmic ray spectrum calculation.

2 Diffusion and Markov stochastic processes

In diffusion models for cosmic ray studies, the distribution function or flux obeys a second-order d-dimensional partial differential equation, which can be in general written as:

$$\frac{\partial u}{\partial t} = \left[\sum_{\mu,\nu}^{d} \frac{\partial}{\partial q_\mu} \left(\frac{1}{2} a_{\mu\nu} \frac{\partial}{\partial q_\nu} - b_\mu \right) + c \right] u + Q \tag{1}$$

where the coordinate $q_\mu(\mu = 1, \cdots d)$ is composed of spatial coordinates and particle momentum or energy, and Q is the source term. Table 1 lists the variables and parameters for studying solar modulation, interstellar propagation and diffusive particle acceleration.

Table 1. Parameters in the diffusion equation for applications to cosmic ray modulation, propagation and acceleration studies.

Parameters	(1) **Heliospheric Modulation**	(2) **Interstellar Propagation**		(3) **Shock Acceleration**
		(A) **Nuclei**	(B) **Electron**	
u	f	N_i (i=species)	N_e	f
q-space	$\mathbf{x}$,	$\mathbf{x}$,	$\mathbf{x}$,	$\mathbf{x}$,
	p	E/amu	E	p
$\frac{1}{2}a_{\mu\nu}$	$\kappa_{\mu\nu}$,	κ	κ	$\kappa_{\mu\nu}$,
	D_p (usually 0)	D_p	D_p	D_p
b_μ	$\mathbf{V} + \mathbf{V}_d$,	usually 0,	0,	$\mathbf{V} + \mathbf{V}_d$,
	$-\frac{1}{3}p\nabla \cdot \mathbf{V}$	b_i	b_e	$-\frac{1}{3}p\nabla \cdot \mathbf{V}$
Q	0 (boundary problem)	continuous	continuous	injection at shock

f – Isotropic distribution function
N_i – Flux per energy range for nuclei
N_e – Flux per energy range for the electron
$\mathbf{x}$ – guiding center position
p – momentum
E – energy
$\kappa_{\mu\nu}$ – diffusion coefficient tensor
κ – diffusion coefficient scalar
D_p – Fermi acceleration coefficient
$\mathbf{V}$ – convection speed of plasma
$\mathbf{V}_d$ – drift speed in magnetic fields
b_i – ionization energy loss rate
b_e – synchrotron energy loss rate
n – interstellar medium density
v – particle speed
σ_i – total cross scalar section for specie i
σ_{ik} – cross section matrix from species k to i
τ_i – radioactive decay time for specie i
τ_{ik} – radioactive decay time matrix from species k to i

The motion of individual particles in diffusion models has always been viewed as random walk since the beginning of theoretical efforts (e.g. Parker, 1965). However, it is only a recent development that the cosmic ray transport equation can be reformulated with stochastic differential equations (Zhang, 1999b). In this approach, the guiding center position of the particle and its momentum (energy) follow a set of Ito stochastic differential equations

$$dq_\mu = \beta_\mu dt + \sum_\sigma \alpha_{\mu,\sigma} dw_\sigma(t) \tag{2}$$

where $w_\sigma(t)$ is a Wiener process (see below) and the sum of σ runs over all required independent random noises. The probability density for the particle in the Markov process determined by (2) to appear in a unit volume at a particular location in q-space at time t, $P(t, q)$, follows the same Fokker-Planck diffusion equation as (1) (Zhang, 1999b) if we let $a_{\mu\nu} = \sum_\sigma \alpha_{\mu,\sigma}\alpha_{\nu,\sigma}$, $\beta_\mu = b_\mu + \frac{1}{2}\sum_\nu \partial a_{\mu\nu}/\partial q_\nu$, and let the process be created at an exponential rate of c as a function of time, i.e. $d(\ln P)/dt = c$. The probability density in q-space can be made proportional to the cosmic ray flux or distribution function. If the probability density starts with a δ-function initially, i.e., the stochastic process starts from a single location point, the solution is the Green's function to the diffusion equation (1); thus the Green's function is often called the transition probability density or propagator. Therefore, stochastic differential equations (2) with an additional creation term can be used to describe diffusion.

Zhang (1999b) applied the Ito stochastic differential equation to studies of modulation, and the results from Monte-Carlo simulation of the stochastic process completely agree with those by directly solving the diffusion equation. One obvious advantage of using the stochastic process approach is that it can reveal the physics of particle diffusion in more detail. For example, we can investigate the trajectory of simulated particles traveling through heliospheric or interstellar magnetic fields and when an ensemble of particles is simulated we can find the distributions of source particles in terms of entry location at the boundary, initial momentum and propagation time (which is approximately proportional to path length). The path length distribution is particularly useful for studies of nuclear fragmentation during interstellar propagation.

3 Path integral representation for the transition probability of Markov processes

For simplicity and also because of space limitation, let us consider only a 1-dimensional stochastic diffusion process governed by an Ito stochastic differential equation

$$dq = \beta(t,q)dt + \alpha(t,q)dw(t) \tag{3}$$

with a creation rate $c(t,q)$. The Wiener process has an associated probability for the process $w(t)$ to transit from w_0 at time t_0 to an interval $w_1 < w(t_1) < w_1 + dw_1$ at t_1 ($t_1 > t_0$):

$$P_1(t_0, w_0; t_1, w_1) = \frac{dw_1}{\sqrt{2\pi(t_1 - t_0)}} \exp -\frac{(w_1 - w_0)^2}{2(t_1 - t_0)} \tag{4}$$

The Wiener process (see Zhang, 1999b) can be understood as the simplest diffusion with a coefficient of $\frac{1}{2}$ and no convection. To calculate the transition probability density for the process described by Equation (3) to get from q_0 at time t_0 to q at t, we normally divide the time interval $\{t_0, t\}$ into N small segments $\{t_0, t_1, t_2, \cdots, t_{N-1}, t_N(t_N = t)\}$. This method is often called discretization. The probability for the process to go through a path $\{q_0, q_1 < q(t_1) < q_1 + dq_1, q_2 < q(t_2) < q_2 + dq_2, \cdots q_{N-1} < q(t_{N-1}) < q_{N-1} + dq_{N-1}, q_N < q(t_N) < q_N + dq_N\}$, during which the driving Wiener process goes through $\{w_0, w_1 < w(t_1) < w_1 + dw_1, w_2 < w(t_2) < w_2 + dw_2, \cdots w_{N-1} < w(t_{N-1}) < w_{N-1} + dw_{N-1}, w_N < w(t_N) < w_N + dw_N\}$, is

$$P_N(\{w_i\}) = \prod_{i=1}^{N} \frac{dw_i}{\sqrt{2\pi\Delta t_i}} \exp -\left[\frac{(\Delta w_i)^2}{2\Delta t_i} - c(t_{i-1}, q_{i-1})\Delta t_i\right], \tag{5}$$

where $\Delta t_i = t_i - t_{i-1}$ and $\Delta w_i = w_i - w_{i-1}$. When $\Delta t_i \to 0$, Δw_i must be $\mathcal{O}(\sqrt{\Delta t_i})$ in order to have non-vanishing probability. The transition probability density from the initial point (t_0, q_0) and the end point (t, q) can be obtained by integrating all the intermediate points, $w_1, w_2, \cdots, w_{N-1}$.

However, the probability density, as obtained directly from (5), is for the w-space. To calculate the probability density in the q-space, we need to find the Jacobian for the transformation to q-coordinates, which can be obtained by finite expansion of the Ito stochastic differential equation (3) to the 6th order (Langouche et al., 1982). Replacing also the argument w_i in the exponential of (5), we obtain a path integral representation for the transition probability density:

$$\begin{aligned} G(t_0, q_0; t, q) \quad = \quad & \frac{1}{\sqrt{2\pi a(t_0, q_0)\Delta t_0}} \int_{q_0=q_0}^{q_N=q} \prod_{i=1}^{N-1} \frac{dq_i}{\sqrt{2\pi a(t_{i-1}, q_{i-1})\Delta t_i}} \exp -\left\{\sum_{i=1}^{N} \frac{1}{2a(t_{i-1}, q_{i-1})}\right. \\ & \left.\left[\frac{\Delta q_i}{\Delta t_i} - \beta(t_{i-1}, q_{i-1})\right]^2 + \frac{\sqrt{a(t_{i-1}, q_{i-1})}}{2} \frac{\partial}{\partial q} \frac{\beta(t_{i-1}, q_{i-1})}{\sqrt{a(t_{i-1}, q_{i-1})}} - c(t_{i-1}, q_{i-1})\right\} \Delta t_i \end{aligned} \tag{6}$$

where $a = \alpha^2$. When we take the limit $N \to \infty$, (6) may be written in a continuous path integral:

$$G(t_0, q_0; t, q) = \int \mathcal{D}q \exp - \int L(t, q, \dot{q})dt \tag{7}$$

where

$$\mathcal{D}q = \frac{1}{\sqrt{2\pi a(t_0, q_0)\Delta t_0}} \lim_{N\to\infty} \prod_{i=1}^{N-1} \frac{dq_i}{\sqrt{2\pi 2a(t_{i-1}, q_{i-1})\Delta t_i}} \tag{8}$$

and the Lagrangian, $L(t, q, \dot{q}), (\dot{q} = \Delta q/\Delta t)$ is

$$L(t, q, \dot{q}) = \frac{1}{2a}[\dot{q} - \beta(t, q)]^2 + \frac{\sqrt{a}}{2}\frac{\partial}{\partial q}\frac{\beta}{\sqrt{a}} - c. \tag{9}$$

The path integral in (7) is consistent with the path integral directly derived from the Fokker-Planck equation (Drozdov, 1993). For higher dimensions, the derivation of the path integral from stochastic differential equations is much more complicated. Interested readers may find rigorous calculations by Langouche et al. (1982)

When the functional integral $\int Ldt$ is minimized, it yields an Euler-Lagrange equation. Thus, the Lagrangian in (9) may be used to find the most probable trajectory for particles.

4 Summary and Application

We have presented a Markov stochastic process approach to the diffusion theory of comic ray modulation, propagation and acceleration. The cosmic ray transport equation is reformulated with the Ito stochastic differential equation. From the stochastic differential equation a Fokker-Planck equation can be derived for the probability density, which is proportional to the cosmic ray flux. The transition probability density of the Markov process is obtained as a path integral consistent with that derived in quantum mechanics.

Figure 1 shows an example of computer calculations of modulated cosmic ray spectra. Three different methods – the path integral approach, stochastic process simulation and numerical method to solve the diffusion equation ("SolMod"; Fisk, 1971) – all agree with each other. In addition to the ability to solve the cosmic ray diffusive transport equation, the two new methods provide the detailed physical processes behind their solutions (Zhang, 1999a, 1999b).

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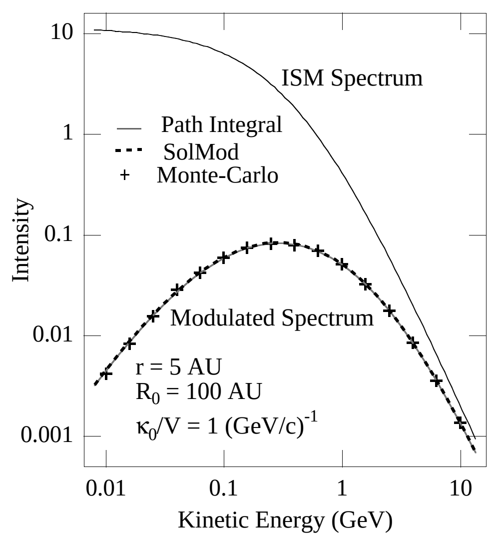

Fig. 1 – Three different calculations of modulated cosmic ray spectra at 5 AU with an input ISM spectrum.

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