Anomalous transport of magnetic field lines

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Abstract

In weak magnetic turbulence, the quasi-linear, diffusive prediction for the spreading of magnetic field lines, as a function of the elapsed distance z along the main magnetic field, $\vec{B}_0$, requires the existence of a sufficiently small correlation length, L_c, that relies upon the flattening of the power spectrum at low wavenumbers. Recent measurements of the solar wind magnetic power spectrum, however, indicate a rather complex spectrum at low frequencies. The transport exponent of the magnetic field lines, α, defined by $\langle \Delta r_\perp^2 \rangle \propto z^\alpha$, will be calculated here, for a general spectrum of turbulence. Normal ($\alpha = 1$), supra- ($\alpha > 1$) and sub-diffusive ($\alpha < 1$) predictions will be made, depending on the lengthscale, z, and the form of the 3D-spectrum. In particular, it will be shown that if the "quasilinear approximation" holds, a diffusion of the magnetic field lines, in the solar wind, is very unlikely below a few AU.

1 Introduction

In both, the interstellar and interplanetary media, is the transport of cosmic rays across the regular component of the magnetic field, for a large part, induced by the transport of the magnetic field lines themselves (Jokipii 1966, 1973; Schlickeiser 1994). The understanding of the behavior of magnetic field lines, in a turbulence composed of random fluctuations, $\delta\vec{B}$, superimposed on a main magnetic field, $\vec{B}_0$, is thus essential to model the propagation of the cosmic rays. The case of "small" magnetic field perturbation is treated by the quasilinear theory (Jokipii & Parker 1968, 1969; Jones 1971), which neglects the perpendicular displacement of the field lines in the derivation of their spreading (first order derivation in $\delta B/B_0$), and predicts their diffusion beyond the parallel correlation length, $L_{c_\parallel}$, defined as the characteristic scale of the two-point correlation function. There is a strong belief among astrophysicists, and physicists in general, that as long as the quasilinear approximation holds, *i.e.*, as long as the perpendicular displacement can be neglected, the quasilinear theory does predict a *diffusion* of the magnetic field lines — *i.e.*, their *linear* spreading across the direction of $\vec{B}_0$, with the distance z along $\vec{B}_0$. However, this diffusive result is conditioned by the existence of a finite correlation length, $L_{c_\parallel}$, small enough to consider the transport of the field lines on much longer scales.

In the initial papers (see references above and Jokipii & Coleman 1968), this correlation length is estimated as the inverse of the upper wavenumber in the low, flat part of the turbulence spectrum. Indeed, a power spectrum flat below $k = L_c^{-1}$ produces a correlation function of the magnetic field perturbation with an exponential cut-off of characteristic scale L_c, and in the 60's, 70's, observations in the solar wind seemed to indicate a flattening of the spectrum below 10^{-5} Hz, which for an Alfvén speed $V_A = 3.10^6$ cm.s^{-1} corresponds to a length scale of 10^{-2} AU, much smaller that the size of the solar system. Recent measurements of the solar-wind magnetic power spectrum, however, do not confirm the spectral flattening at such a high frequency. In fact, the spectrum at low frequencies is very much influenced by the solar rotation, and it is difficult to infer from the measured frequency spectrum, what the $\vec{k}$-spectrum, projected along the main magnetic field, at one time, and along one flux tube — which is the one needed to derive the field lines spreading —, really is. But the strong anisotropy of the spectrum, observed in the solar wind (Ghosh et al. 1998 — see also references therein), seem to privilege an extended slope of the projected $k_{||}$-spectrum, down to very low wavenumbers — the projection of a 3D-spectrum is flat below the injection scale when the spectrum is isotropic, but not if it is anisotropic (see, for instance, Ragot ?). In this case of extended, non-flat projected spectrum, a study of the field lines transport is still needed, even in the quasilinear regime of magnetic field perturbation.

We will briefly present here, *in the quasilinear regime*, the calculation of the field lines spreading, $\langle \Delta x^2 \rangle$, across $\vec{B}_0$, as a function of the length scale, z, and the form of the 3D-spectrum of the magnetic turbulence. We will deduce from this calculation the transport exponent, α, defined by $\langle \Delta x^2 \rangle \propto z^\alpha$ (*e.g.*, Ragot & Kirk 1997), and show that a decreasing power towards the increasing wavenumbers produces a supradiffusion ($\alpha > 1$) of the field lines, whereas a increasing power (inverted spectrum) yields a subdiffusion ($\alpha < 1$), as long as the spectral index is not too high in absolute value. Finally, we will apply these results to the solar wind, where we will assume that the quasilinear approximation holds, which might be true far from the poles.

2 Field lines spreading and transport exponent

If the perpendicular deviation is neglected — "quasilinear approximation" —, the displacement, along the axis x, of the field line that goes through the point $\vec{r}_0 = (x_0, y_0, z_0)$ can be written as

$$\Delta x = x(\vec{r}_0, z) - x_0 = \int_{z_0}^{z} b_x(x_0, y_0, z')dz' \ , \tag{1}$$

where $\vec{b}$ stands for $\delta\vec{B}/B_0$, so that the variance $\langle \Delta x^2 \rangle$, where $\langle\ \rangle$ denotes an average over a statistical ensemble of systems, can be expressed as:

$$\langle \Delta x^2 \rangle = \int_{z_0}^{z} dz' \int_{z_0}^{z} dz'' \, \langle b_x(x_0, y_0, z') b_x(x_0, y_0, z'') \rangle = 2\Delta z \int_0^{\Delta z} ds \left(1 - \frac{s}{\Delta z}\right) R_{xx}(s) \ . \tag{2}$$

$R_{xx}(s) = \langle b_x(x_0, y_0, z_0) b_x(x_0, y_0, z_0 + s) \rangle$ is the two-point correlation function of the magnetic field along x. In the usual quasilinear theory, R_{xx} is assumed to "cut-off" on the length scale $L_{c_{||}}$ — parallel correlation length —, and the limit $\Delta z \gg L_{c_{||}}$ is taken, so that

$$\frac{\langle \Delta x^2 \rangle}{2\Delta z} = \int_0^{+\infty} ds \, R_{xx}(s) \equiv D \ . \tag{3}$$

It shows that field lines diffuse on length scales much longer than $L_{c_{||}}$, with the diffusion coefficient D, but it does not prove that $L_{c_{||}}$ exists, and is very much smaller than the size of the system, which is just necessary to observe a diffusion *in* the system.

Dropping the assumption concerning the existence of a finite correlation length $L_{c_{||}}$, we calculate below the field lines spreading for a general power spectrum of turbulence. From

$$b_x(\vec{r}) = 2 \int d\vec{k}_\perp \int_0^{k_M} dk_{||} \, \tilde{b}_x(\vec{k}) \cos\left(\vec{k}_\perp \cdot \vec{r} + k_{||} z + \phi_{\vec{k}}\right) \ , \tag{4}$$

$\tilde{b}_x(\vec{k}) e^{i\phi_{\vec{k}}}$ being the Fourier transform of $b_x(\vec{r})$ — $\tilde{b}_x(\vec{k}) > 0$ —, and k_M the largest wavenumber in the spectrum, we can show that:

$$\langle (x - x_0)^2 \rangle = 2k_m^3 \int_{z_0}^{z} dz' \int_{z_0}^{z} dz'' \int d\vec{k} \, b_x^2(\vec{k}) \cos(k_{||}(z' - z'')) \ , \tag{5}$$

where k_m is the minimum wavenumber — the inverse of the system' size. To derive Eq. (5), we assumed, as in the quasilinear theory, that the phases $\phi_{\vec{k}}$ decorrelate on the scale k_m — no spectral structuring. (This assumption could be dropped by introducing a different phase-correlation scale.) Integrating over z' and z'', we obtain:

$$\langle (x - x_0)^2 \rangle = 4k_m^3 \int_0^{k_M} dk_{||} \left[1 - \cos\left(k_{||}(z - z_0)\right)\right] k_{||}^{-2} P_{x_{||}}(k_{||}) \ , \tag{6}$$

$$\text{where} \qquad P_{x_{||}}(k_{||}) = \int_0^{2\pi} d\phi \int_{\sqrt{\max(0, k_m^2 - k_{||}^2)}}^{\sqrt{k_M^2 - k_{||}^2}} dk_\perp \, k_\perp b_x^2(k_{||}, k_\perp, \phi) \tag{7}$$

is the x-component of the power spectrum projected along $\vec{B}_0$.

If the projected spectrum $P_{x_\parallel}(k_\parallel)$ is smooth enough to be represented as a series of power laws, integrating in the complex plane, we can write the spreading of the field lines as follows:

$$\langle (x-x_0)^2 \rangle = 4k_m^3 \sum_{j=1}^{J-1} \frac{P_{x_\parallel}(k_j)}{k_j^{-a_j}} \left\{ \frac{k_{j+1}^{-a_j-1} - k_j^{-a_j-1}}{-a_j-1} \right.$$

$$\left. -Z^{a_j+1} \Re \left[e^{i\frac{\pi}{2}(a_j+1)} \left(\Gamma(-a_j-1, ik_jZ) - \Gamma(-a_j-1, ik_{j+1}Z) \right) \right] \right\} , \qquad (8)$$

where $\Re[x]$ denotes the real part of x, and $\Gamma(a,x)$ the incomplete Gamma function. $Z = z - z_0$, and the parallel wavenumbers k_1, k_2, ... k_J are chosen so that there is a well defined spectral index, $-a_j$, between k_j and k_{j+1} — J can be arbitrarily large, to allow for an accurate description of $P_{x_\parallel}(k_\parallel)$. Provided the power spectrum is not overwhelmed by the very low $k_\parallel$ — corresponding to a dominant, perpendicular 2D-turbulence with a "divergence" in $k_\parallel = 0$ —, choosing k_1^{-1} much larger than the largest relevant scale of the system guarantees a negligeable contribution from the interval $[0, k_1[$ of the spectrum (cf Ragot ?).

From Eq. (8), we can now estimate the transport exponent:

$$\alpha = \frac{d \log \langle (x-x_0)^2 \rangle}{d \log Z} , \qquad (9)$$

as a function of the spectral shape, and the elapsed distance Z along $\vec{B}_0$. To learn about the relation, spectral index of the turbulence - transport exponent, we will first assume a simple power law $-a$ for the turbulence spectrum, bound by k_1 and k_2.

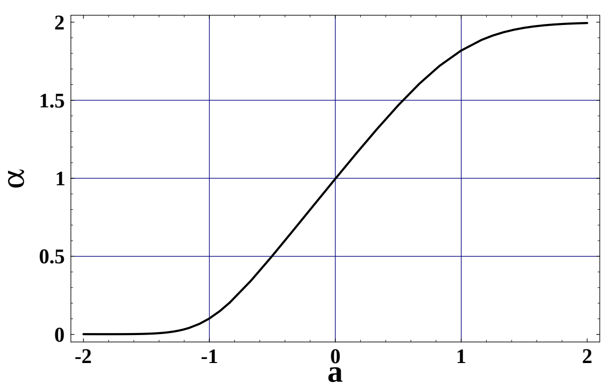

Figure 1: *Transport exponent α as a function of the spectral index a, for $\log((k_2 - k_1)/k_1) = 6$ and $\log(k_1 Z) = -2$. For $a < -1$, α is averaged on Z.*

Figure 1 presents, as a function of a, the transport exponent α taken at a distance Z such that $k_2^{-1} \ll Z \ll k_1^{-1}$. Clearly, the spreading of the field lines is only diffusive when the spectrum is flat — and broad, which can be seen by plotting $\langle (x-x_0)^2 \rangle$ as a function of $k_2 - k_1$ and Z (cf Ragot ?). For all decreasing power laws, the field lines supradiffuse ($\alpha > 1$), whereas for inverted power laws, the field lines subdiffuse ($\alpha < 1$), as long as $a > -2$. — Below $a = -1$, which is already quite unreallistic for a turbulence spectrum, α does not converge to a unique value, but if averaged on a broad range of length scales Z, tends to 0. — The deviation of the transport exponent from its diffusive value 1 already exceeds 0.5 for a spectral index of 0.6, in absolute value.

3 Application to the solar wind and conclusion

We now apply relation (8) to the solar wind turbulence. For this we choose projected spectra, in $k_\parallel$, which correspond to the observed frequency spectrum (Goldstein et al. 1995) in the range of frequencies where: the observations exist, and are not too much influenced by the solar rotation, *i.e.*, above $\approx 10^{-6}$ Hz. Below, we make the spectrum flatten in different ways (see Fig. 2). The case which should produce the earliest diffusive spreading is given by a flat projected spectrum below 4.10^{-7} Hz (continuous line). It appears that already in this case, α does not reach 1 before a few AU. The transport exponent obtained by only including frequencies above 10^{-8} Hz is plotted in dashed line. It indicates which maximal scale could effectively be studied on the basis of the current observations — despite the very uncertain equivalence between the observed frequency spectrum and $P_{x_\parallel}(k_\parallel)$. The dot-dashed curve, which only differs from the continuous curve by a slight bump in the spectrum around 4.10^{-7} Hz, shows how sensitive to the sprectrum, the spreading of the field lines is. Finally, we can not exclude that the spectral index remains slightly negative further down (*e.g.*, down to 10^{-10} Hz). If so, the transport of the field lines remains supradiffusive on all the relevant scales of the system.

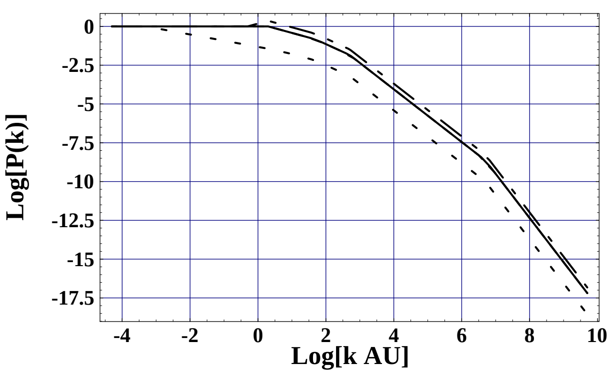

Figure 2: *Projected spectra $P_{x_{||}}(k_{||})$ used to compute the transport exponents in Figure 3. The spectral index $-a$ is: -0.6 between 4.10^{-7} Hz and 10^{-6} Hz, -1 from 10^{-6} Hz to 10^{-4} Hz, -1.7 up to 1 Hz and -2.85 above. The continuous spectrum is flat between 10^{-11} Hz and 4.10^{-7} Hz. The dot-dashed spectrum has a slight bump — inverted spectrum in $k^{0.6}$ — between 10^{-7} Hz and 4.10^{-7} Hz, and is flat below, down to 10^{-11} Hz. Between 10^{-6} Hz and 10^{-10} Hz, the spectral index is -0.4 for the dot curve.*

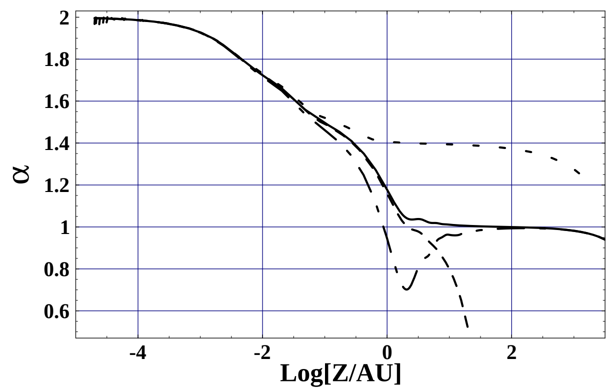

Figure 3: *Transport exponent α, as a function of the logarithm of the distance Z, in astronomical units, for the different models of solar wind turbulence which projected spectra are presented in Figure 2. The line styles are identical in both figures. The spectrum corresponding to the long-dashed line is identical to the one in continuous line down to 10^{-8} Hz, and cuts off below. Above 10^{-7} Hz (~ 0.5 AU), the spectra are based on the observations by Goldstein et al. (1995). $V_A = 3.10^6$ cm.s^{-1}.*

In the quasilinear regime of magnetic field turbulence, we have shown that to a positive spectral index a of the turbulence, corresponds a supradiffusion ($\alpha > 1$) of the magnetic field lines. For an inverted spectrum, the transport remains subdiffusive ($\alpha < 1$) as long as $a > -2$. Using recent observations of the solar wind turbulence spectrum, we have found that the spreading of the magnetic field lines, if the "quasilinear approximation" holds, cannot be diffusive before a few AU in the heliosphere.

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