29th International Cosmic Ray Conference Pune (2005) **00**, 101–104

Molière lateral distribution with ionization for fast charged particles

Takao Nakatsuka[a] and Kazuhide Okei[b]

(a) Okayama Shoka University, Tsushima-Kyomachi, Okayama 700-8601, Japan
(b) Dept. of Physics, Okayama University, Okayama 700-8530, Japan
Presenter: T. Nakatsuka (nakatuka@osu.ac.jp), jap-nakatsuka-T-abs1-he12-oral

Molière's analytical result for distribution of arbitrary linear combination between the deflection angle and the lateral displacement is improved to take account ionization loss. Analytical results of the lateral distribution for fast charged particles after penetrating through matters, as one of the combinations, are compared with our Monte Carlo simulation results using the screened Rutherford cross-section. The results will be important for designings and analyses of experiment and simulation concerning charged particles.

1. Introduction

It is very important for us cosmic-ray physicists to investigate transport properties of charged particles traversing through materials. Molière and Bethe have found very accurate results of angular distribution [1, 2, 3], as well as distribution for arbitrary linear combination between the deflection angle $\vec{\theta}$ and the lateral displacement $\vec{r}$ [4], both under the fixed-energy condition. We have succeeded in improving the Molière-Bethe results for the angular distribution to take account ionization loss [5], by using Nishimura-Kamata formulation of the theory [6, 7]. We propose improved Molière results of distribution for arbitrary linear combination between the deflection angle $\vec{\theta}$ and the lateral displacement $\vec{r}$, taking account of ionization loss.

2. Molière distributions with ionization for linear combination of the deflection angle and the lateral displacement

Let the simultaneous distribution between the two be $f(\vec{\theta},\vec{r},t)d\vec{\theta}d\vec{r}$ and its Fourier transforms be $\tilde{f}(\vec{\zeta},\vec{\eta},t)$:

$$f(\vec{\theta},\vec{r},t) = \frac{1}{4\pi^2}\iiiint e^{-i\vec{\theta}\vec{\zeta}-i\vec{r}\vec{\eta}}\tilde{f}(\vec{\zeta},\vec{\eta},t)d\vec{\zeta}d\vec{\eta}, \tag{1}$$

where $\vec{\zeta}$ and $\vec{\eta}$ denote the Fourier variables corresponding to $\vec{\theta}$ and $\vec{r}$, respectively. The diffusion equation for Fourier transforms of the simultaneous distribution is described as

$$\frac{\partial \tilde{f}}{\partial t'} = \vec{\eta}\frac{\partial \tilde{f}}{\partial \vec{\zeta}'} - \frac{K^2\zeta'^2}{4E'^2}\tilde{f}\{1-\frac{1}{\Omega}\ln\frac{K^2\zeta'^2}{4E'^2}\} \tag{2}$$

under the Kamata-Nishimura formulation, where the variables $\vec{\zeta}'$ and E' are primed to note that they change together with the differential variable t'. Then Eq. (2) can be integrated as

$$\ln 4\pi^2\tilde{f} \quad = \quad \int_0^1 \frac{K^2t(\vec{\zeta}+\vec{\eta}tu)^2}{4\Omega(E+\varepsilon tu)^2}\ln\frac{K^2(\vec{\zeta}+\vec{\eta}tu)^2}{4e^{\Omega}(E+\varepsilon tu)^2}du, \tag{3}$$

where E denotes the destination energy.

Let $\vec{\rho}$ be the linear combination of the deflection angle $\vec{\theta}$ and the lateral displacement $\vec{r}$, or the chord angle $\vec{r}/t$ [4, 8], with respective weights of a and b,

$$\vec{\rho} = a\vec{\theta} + b\vec{r}/t, \tag{4}$$

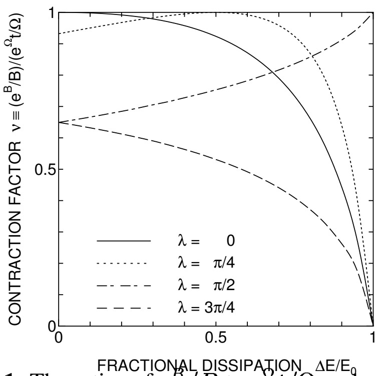

Figure 1. The ratios of e^B/B to $e^\Omega t/\Omega$ under the ionization process, i.e. the contraction factors ν for distributions of the linear combination $\vec{\rho} \equiv \vec{\theta}\cos\lambda + (\vec{r}/t)\sin\lambda$, defined for the extreme-relativistic charged particles. Abscissa denotes the fraction of dissipated energy $(E_0 - E)/E_0$.

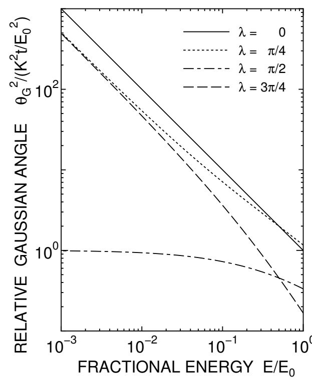

Figure 2. The ratios of $\theta_{\rm G}^2$, i.e. the gaussian mean square angle for distributions of the linear combination angle $\vec{\rho} \equiv \vec{\theta}\cos\lambda + (\vec{r}/t)\sin\lambda$ under the ionization process defined for extreme-relativistic charged particles, to $\Omega^{-1}K^2t/E_0^2$. Abscissa denotes the fraction of residual energy E/E_0.

and be its probability density, then we have

$$g(\vec{\rho},t)d\vec{\rho} = \frac{td\vec{\rho}}{b}\iint f(\vec{\theta},\frac{t}{b}(\vec{\rho}-a\vec{\theta}),t)d\vec{\theta}. \tag{5}$$

Let $\tilde{g}(\vec{\xi},t)$ be the Fourier transforms of $g(\vec{\rho},t)$,

$$\tilde{g}(\vec{\xi}) = \frac{1}{2\pi}\iint e^{-i\vec{\xi}\vec{\rho}}g(\vec{\rho},t)d\vec{\rho} = 2\pi\tilde{f}(a\vec{\xi},b\vec{\xi}/t,t), \tag{6}$$

then we have

$$\begin{aligned}
\ln 2\pi\tilde{g} &= \int_0^1 \frac{K^2t\xi^2(a+bu)^2}{4\Omega(E+\varepsilon tu)^2}\ln\frac{K^2\xi^2(a+bu)^2}{4e^\Omega(E+\varepsilon tu)^2}du \\
&= \int_E^{E_0}\frac{b^2K^2t\xi^2}{4\Omega\varepsilon^3t^3}(1-\frac{Q}{E'})^2\ln[\frac{b^2K^2\xi^2}{4e^\Omega\varepsilon^2t^2}(1-\frac{Q}{E'})^2]dE' \\
&= \int_{Q/E_0}^{Q/E}\frac{b^2QK^2t\xi^2}{4\Omega\varepsilon^3t^3}(1-\frac{1}{s})^2\ln\frac{b^2K^2\xi^2(1-s)^2}{4e^\Omega\varepsilon^2t^2}ds \\
&= -\frac{b^2QK^2t\xi^2}{4\Omega(E_0-E)^3}\{2[s+\ln|s|-(s-\frac{1}{s})\ln|1-s|-2\mathcal{L}_2(s)]_{Q/E_0}^{Q/E} \\
&\qquad -[s-\frac{1}{s}-2\ln|s|]_{Q/E_0}^{Q/E}\ln\frac{b^2K^2\xi^2}{4e^\Omega(E_0-E)^2}\},
\end{aligned} \tag{7}$$

where we introduced an energy parameter Q and the dilogarithm function $\mathcal{L}_2(z)$ [9, 10],

$$Q \equiv E - (a/b)\varepsilon t, \tag{8}$$

$$\mathcal{L}_2(z) = \int_z^0 \frac{\ln|1-t|}{t}dt. \tag{9}$$

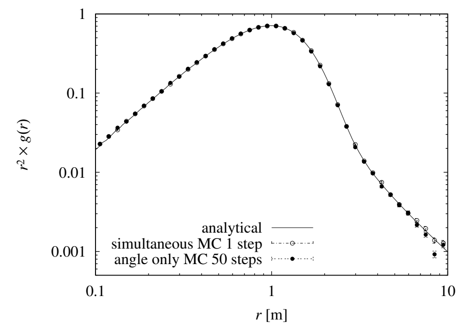

Figure 3. Analytical prediction of lateral distribution of muon after dissipating half of its energy agrees with the Monte Carlo result.

Thus we can get the spatial distribution for $\vec{\rho}$ by using our translation formula [11]:

$$g(\vec{\rho}, t)d\vec{\rho} = f(\vartheta)d\vec{\vartheta} \quad \text{with} \quad \vec{\vartheta} = \vec{\rho}/\theta_{\mathrm{M}}, \tag{10}$$

where

$$2\pi f(\vartheta) = f^{(0)}(\vartheta) + B^{-1}f^{(1)}(\vartheta) + B^{-2}f^{(2)}(\vartheta) + \dots, \tag{11}$$

and

$$B - \ln B = \Omega - \ln\Omega + \ln\{(Q/\varepsilon)[s - s^{-1} - 2\ln|s|]^{Q/E}_{Q/E_0}\} \\ + \frac{2[s + \ln|s| - (s - s^{-1})\ln|1 - s| - 2\mathcal{L}_2(s)]^{Q/E}_{Q/E_0}}{[s - s^{-1} - 2\ln|s|]^{Q/E}_{Q/E_0}}, \tag{12}$$

$$\theta^2_{\mathrm{M}} = \frac{B}{\Omega}\theta^2_{\mathrm{G}} \quad \text{with} \quad \theta^2_{\mathrm{G}} \equiv \frac{b^2QK^2t}{(E_0 - E)^3}[s - \frac{1}{s} - 2\ln|s|]^{Q/E}_{Q/E_0}. \tag{13}$$

θ^2_{G} denotes the gaussian mean square angle for the combination angle of Eq. (4). The projected distribution for $\vec{\rho}$ can also be derived likewise, The limits of Eqs. (12) and (13) at $\varepsilon \to 0$ give the Molière's results for the combination angle under the fixed-energy condition [4].

3. Results and discussions

We compare the value of e^B/B derived from Eq. (12) with $e^{\Omega}t/\Omega$ as is e^B/B of the angular distribution under the fixed-energy process. The ratio $(e^B/B)/(e^{\Omega}t/\Omega)$, which should be the definition of the contraction factor ν for the linear combination $\vec{\rho}$,

$$\nu \equiv (B^{-1}e^B)/(\Omega^{-1}e^{\Omega}t), \tag{14}$$

is indicated in Table 1 and Fig. 1, taking

$$a = \cos\lambda \quad \text{and} \quad b = \sin\lambda \tag{15}$$

as Molière did [4]. We also compare the mean square gaussian angle θ^2_{G} of Eq. (13) for the combination angle $\vec{\rho}$ with θ^2_{G} for the deflection angle under the fixed-energy condition K^2t/E_0^2. The ratio $\theta^2_{\mathrm{G}}/(K^2t/E_0^2)$ is

Table 1. The contraction factor ν for the linear combination $\vec{\rho}$, taking $a = \cos\lambda$ and $b = \sin\lambda$.

$\Delta E/E_0$	$\lambda = 0$	$\lambda = \pi/4$	$\lambda = \pi/2$	$\lambda = 3\pi/4$
.00	1.000E+00	9.320E-01	6.492E-01	6.492E-01
.05	9.996E-01	9.412E-01	6.575E-01	6.409E-01
.10	9.982E-01	9.502E-01	6.662E-01	6.320E-01
.15	9.956E-01	9.590E-01	6.752E-01	6.225E-01
.20	9.917E-01	9.677E-01	6.846E-01	6.124E-01
.25	9.863E-01	9.760E-01	6.945E-01	6.016E-01
.30	9.791E-01	9.829E-01	7.054E-01	5.902E-01
.35	9.696E-01	9.895E-01	7.157E-01	5.772E-01
.40	9.576E-01	9.949E-01	7.278E-01	5.632E-01
.45	9.425E-01	9.986E-01	7.400E-01	5.481E-01
.50	9.236E-01	1.000E+00	7.531E-01	5.310E-01
.55	9.002E-01	9.982E-01	7.670E-01	5.126E-01
.60	8.711E-01	9.921E-01	7.821E-01	4.917E-01
.65	8.349E-01	9.797E-01	7.986E-01	4.680E-01
.70	7.898E-01	9.583E-01	8.164E-01	4.409E-01
.75	7.331E-01	9.236E-01	8.360E-01	4.091E-01
.80	6.609E-01	8.686E-01	8.580E-01	3.711E-01
.85	5.674E-01	7.811E-01	8.829E-01	3.240E-01
.90	4.430E-01	6.390E-01	9.122E-01	2.625E-01
.95	2.695E-01	3.986E-01	9.482E-01	1.740E-01
1.00	.000E+00	.000E+00	1.000E+00	.000E+00

indicated in Fig. 2. The both ratios are functions of the fraction of residual energy E/E_0, so that functions of the fraction of dissipated energy $\varepsilon t/E_0$. Our ratios at $\Delta E/E_0 = 0$ in Fig. 1 and ours at $E/E_0 = 1$ in Fig. 2 agree with Molière's results without ionization in his Table 1 [4]. We find only the chord-angle distribution or the lateral distribution ($\lambda = \pi/2$) has the ratios of non-zero finite value at the limit of $E \to 0$.

Molière lateral distribution with ionization is derived as the combination of $a = 0$ and $b = 1$, or $\lambda = \pi/2$. The result for 100 GeV muon, having dissipated half of its energy after penetrating water of 250 m, is compared with the simulation result by our high-accurate and high efficient Monte Carlo code [12, 13] in Fig. 3.

Under the special case of $a/b = E/(\varepsilon t)$, including the case of $\lambda = \pi/2$ with $E \to 0$, Eq. (7) can be easily integrated as

$$\ln 2\pi\tilde{g} = \frac{(a+b)^2 K^2 t\xi^2}{4\Omega E_0^2} \ln \frac{(a+b)^2 K^2 t\xi^2}{4e^{\Omega} E_0^2}, \tag{16}$$

so we find from our translation formula [11] the distribution for linear combination $\vec{\rho}$ under this condition has the same Molière distribution as the angular distribution without ionization loss, only with the $(a+b)$-times large/small scale angle θ_{M}.

4. Conclusions

Molière distribution with ionization for any linear combination between the deflection angle $\vec{\theta}$ and the lateral displacement $\vec{r}$ is solved analytically, using Nishimura-Kamata formulation of the theory. Derived lateral distribution agrees very well with our Monte Carlo result. The theory will be helpful for our designing and analyses of experiments concerning charged particles.

5. Acknowledgements

We are very indebted to Prof. Jun Nishimura for useful discussions and many advices.

References

[1] G. Molière, Z. Naturforsch. **2a**, 133(1947).
[2] G. Molière, Z. Naturforsch. **3a**, 78(1948).
[3] H.A. Bethe, Phys. Rev. **89**, 1256(1953).
[4] G. Molière, Z. Naturforsch. **10a**, 177(1955).
[5] T. Nakatsuka, *Proc. of the 26th ICRC*, Salt Lake City, **1**, 486(1999).
[6] K. Kamata and J. Nishimura, Prog. Theor. Phys. Suppl. **6**, 93(1958).
[7] J. Nishimura, in *Handbuch der Physik, Band 46*, edited by S. Flügge (Springer, Berlin, 1967), Teil **2**, p. 1.
[8] W.T. Scott, Rev. Mod. Phys. **35**, 231(1963).
[9] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), Chap. 27.
[10] W. Gröbner and N. Hofreiter, *Integraltafeln* (Springer, Wien, 1950), **I**.
[11] T. Nakatsuka, *Proc. of the 26th ICRC*, Salt Lake City, **1**, 522(1999).
[12] K. Okei and T. Nakatsuka, *Proc. of the 28th ICRC*, Tsukuba, **3**, 1499(2003).
[13] K. Okei and T. Nakatsuka, *Proc. of the 29th ICRC*, Pune, in this conference.