shadow of the sphere plays no significant role in this case as the points of interest are far from the sphere. In the second case the shadow of the sphere is important as the points of interest are close to the boundary. In the limit, the region $c>a$ for $\mu>0$ disappears. The distribution function at points near the sphere may be discussed from the point of view of this approximation.

Because of the existence of the shadow and the discontinuity in ψ, measurements of the critical angle θ_r and Δ can, in principle, determine the radius a of the sphere and the mean free path λ of the neutrons. If it is supposed that μ_r is determined at two positions along a radius so that $\delta r = r_2 - r_1$ is known, then, from (16),

$$a^2 = r_1^2(1-\mu_{r1}^2) = r_2^2(1-\mu_{r2}^2).$$

There are three equations for the three unknowns a, r_1, and r_2. In (15), r is expressed in terms of λ as the unit of length. We then write, from (15),

$$\ln\Delta = -(1/\lambda)r\mu_r + \ln F.$$

The slope of the line obtained from a plot of $\ln\Delta$ against $r\mu_r$ determines λ. The quantity Δ may be measured in an arbitrary unit and the quantity $r\mu_r$ is provided by the measurements of μ_r and δr.

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Angular Correlation of Scattered Annihilation Radiation*

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If the two photons emitted in an annihilation process are scattered, their initial cross-polarization leads to an angular correlation of the scattered radiation. This correlation effect is calculated, and yields a substantial azimuthal asymmetry. It is shown that one may regard the scattering of one photon as performing a partial analysis of the polarization of the other photon.

1. INTRODUCTION

According to pair theory[1] the dominant type of annihilation is one in which the positron-electron pair has zero relative angular momentum. Associated with this is the cross-polarization of the two quanta emitted in the annihilation process. If one photon is linearly polarized in one plane, the other photon, which goes off in the opposite direction, is linearly polarized in the perpendicular plane. A similar relation exists for any state of polarization of one photon.

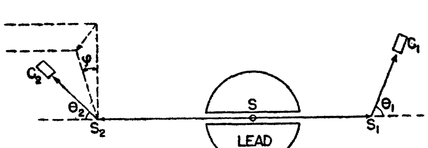

FIG. 1. Schematic diagram of experimental arrangement.

Wheeler[2] has suggested an experiment to test this prediction, involving coincidence measurements of the scattering of both of the annihilation photons. The arrangement is represented schematically in Fig. 1.

A source S of annihilation radiation (a radioactive source of slow positrons covered with a foil) is placed at the center of a lead sphere with a narrow channel drilled through it. The photons, each of energy mc^2, passing through the channel are scattered by scatterers S_1 and S_2 and recorded by gamma-ray counters C_1 and C_2. Coincidences between the two counters are recorded when the azimuths of the two counters are identical ($\varphi=0$) and when the azimuths differ by a right

* Research carried out at the Brookhaven National Laboratory under the auspices of the Atomic Energy Commission.

[1] P. A. M. Dirac, Proc. Camb. Phil. Soc. 26, 361 (1930).

[2] J. A. Wheeler, Ann. N. Y. Acad. Sci. 48, 219 (1946).

angle ($\varphi=\pi/2$), and the ratio is determined. According to Wheeler, the calculated ratio ($N_{\varphi=\pi/2}/N_{\varphi=0}$) for the case of ideal geometry is 1.080 when the scattering angles θ_1 and θ_2 are 90°; and the theoretically most favorable ratio of 1.100 is obtained when the scattering angles are reduced to 74°30′.

The theory of this proposed experiment has been re-examined, using two different approaches, and results different from those of Wheeler are obtained.[3] The following ratio is obtained for the case in which the two photons are scattered through the same angle θ:

$$\rho=\frac{N_{\varphi=\pi/2}}{N_{\varphi=0}}=1+\frac{2\sin^4\theta}{\gamma^2-2\gamma\sin^2\theta};$$
$$\gamma=2-\cos\theta+\frac{1}{2-\cos\theta}. \quad (1)$$

For scattering angles of 90° the ratio reduces to 2.60. The maximum ratio of 2.85 is obtained for scattering angles of 82°. The asymmetry ratio ρ is plotted as a function of θ in Fig. 2.

2. PARTIAL POLARIZATION ANALYSIS METHOD

One approach to the problem is to view the scattering of one photon as performing a partial analysis of the polarization of the other photon. That is, the observation of one scattered photon gives information about the initial state of polarization of the other photon.

The statement that the two light quanta are polarized at right angles to each other is readily expressed in terms of the Schroedinger wave functionals. Let $\psi(\cdots, N_{\mathbf{k}\lambda}, \cdots)$ denote the wave functional for the light quanta when there are $N_{\mathbf{k}\lambda}$ quanta, having momenta $\mathbf{k}$ and polarization indicated by the index λ. The annihilation radiation consisting of two quanta having momenta $\mathbf{k}_0$ and $-\mathbf{k}_0$ may then be described by the following wave functional

$$\frac{1}{\sqrt{2}}\{\psi(\cdots 1_{\mathbf{k}_01}, 0_{\mathbf{k}_02}, \cdots 0_{-\mathbf{k}_01}, 1_{-\mathbf{k}_02}, \cdots)$$
$$+\psi(\cdots 0_{\mathbf{k}_01}, 1_{\mathbf{k}_02}, \cdots 1_{-\mathbf{k}_01}, 0_{-\mathbf{k}_02}, \cdots)\}, \quad (2)$$

with arguments of 0 for all quanta not explicitly indicated. The polarization indices 1 and 2 for $\mathbf{k}_0$ and $-\mathbf{k}_0$ refer to the plane-polarization directions as shown in Fig. 3, in which the $\boldsymbol{\varepsilon}$'s are unit vectors.

For the above expression to represent correctly the fact that the quanta with the momenta $\mathbf{k}_0$

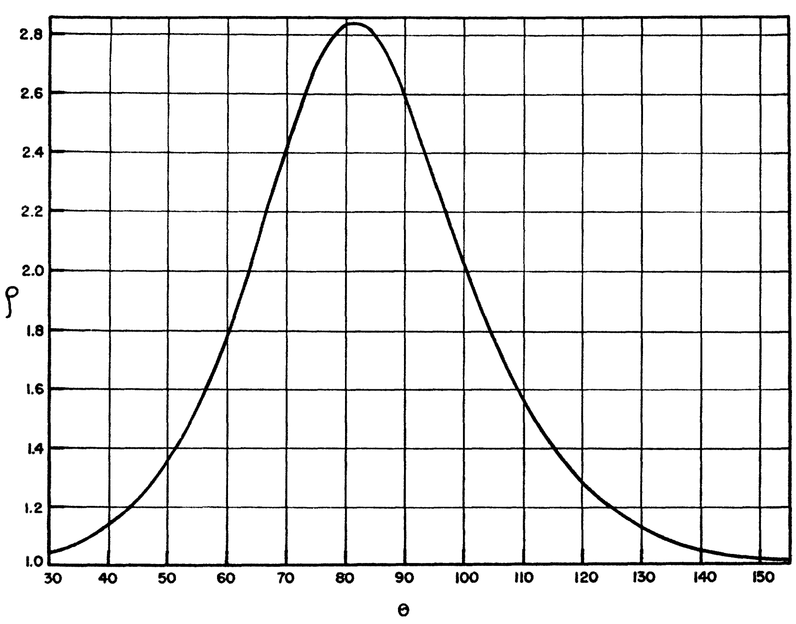

FIG. 2. Asymmetry of coincidence counting rate, ρ, for ideal geometry, as a function of the scattering angle θ.

[3] Since these results were obtained there has appeared a brief article by M. H. L. Pryce and J. C. Ward (Nature 160, 435 (Sept. 27, 1947)) in which the result of similar calculations is reported. Their formula agrees with the one obtained here (Eq. (19)).

and $-\mathbf{k}_0$ are polarized at right angles to each other, the phases of the wave functionals are chosen so that $\psi(\cdots 1_{\mathbf{k}_{01}}, 0_{\mathbf{k}_{02}}, \cdots)\cos\mu + \psi(\cdots 0_{\mathbf{k}_{01}}, 1_{\mathbf{k}_{02}}, \cdots)\sin\mu$ represents a plane-polarized quantum making an angle μ with the $\boldsymbol{\varepsilon}_{\mathbf{k}_{01}}$ direction, and an angle $\pi/2-\mu$ with the $\boldsymbol{\varepsilon}_{\mathbf{k}_{02}}$ direction for all momenta $\mathbf{k}_0$. It may now be easily verified for a new system of axes for the resolution of the polarization of the light quanta,

$$\boldsymbol{\varepsilon}_{\mathbf{k}_{01}'} = \boldsymbol{\varepsilon}_{-\mathbf{k}_{01}'} = \boldsymbol{\varepsilon}_{\mathbf{k}_{01}}\cos\mu + \boldsymbol{\varepsilon}_{\mathbf{k}_{02}}\sin\mu,$$
$$\boldsymbol{\varepsilon}_{\mathbf{k}_{02}'} = -\boldsymbol{\varepsilon}_{-\mathbf{k}_{02}'} = -\boldsymbol{\varepsilon}_{\mathbf{k}_{01}}\sin\mu + \boldsymbol{\varepsilon}_{\mathbf{k}_{02}}\cos\mu,$$

that the wave functional above becomes

$$\frac{1}{\sqrt{2}}\{\psi(\cdots 1_{\mathbf{k}_{01}'}, 0_{\mathbf{k}_{02}'}, \cdots 0_{-\mathbf{k}_{01}'}, 1_{-\mathbf{k}_{02}'}, \cdots) + \psi(\cdots 0_{\mathbf{k}_{01}'}, 1_{\mathbf{k}_{02}'}, \cdots 1_{-\mathbf{k}_{01}'}, 0_{-\mathbf{k}_{02}'}, \cdots)\}.$$

It is now clear on inspection that this wave functional represents two light quanta with opposite momenta; each quantum is unpolarized, but the two quanta are polarized at right angles to each other.

We now consider the manner in which the scattering of a light quantum by an electron partially analyzes the polarization of the original quantum. According to the Klein-Nishina formula[4] the polarization dependence of the differential scattering cross section for a photon of momentum $\mathbf{k}_0$ scattered into a photon of momentum $\mathbf{k}$ is given by the factor $k/k_0+k_0/k-2+4\cos^2\Theta$, in which Θ is the angle between the direction of polarization of the incident quantum and the direction of polarization of the scattered quantum. If we average this over the polarization directions of the scattered quantum we find for the polarization dependence of the initial quantum the factor

$$\frac{k}{k_0}+\frac{k_0}{k}-2\sin^2\theta\cos^2\phi, \qquad (3)$$

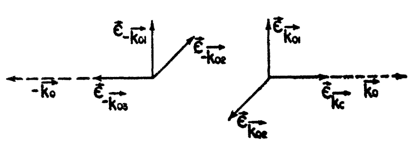

FIG. 3. Coordinate systems for representing polarization directions of the initial quanta. *Note added in proof:* $\mathbf{k}_0$, the subscript to the first unit vector from the right, should read $\mathbf{k}_{03}$.

[4] O. Klein and Y. Nishina, Zeits. f. Physik **52**, 853 (1929); Y. Nishina, *ibid.* **52**, 869 (1929).

in which θ is the angle of scattering and ϕ is the angle between the plane of the scattering and the direction of polarization of the incident quantum. For a quantum polarized in the plane of scattering (3) gives us the factor $k/k_0+k_0/k-2\sin^2\theta$; for a quantum polarized perpendicular to the plane of the scattering we obtain $k/k_0+k_0/k$. If the quantum is polarized making an angle ϕ with the plane of scattering, then the probability that it is polarized in the plane of the scattering is $\cos^2\phi$ and the probability that it is polarized at right angles to the plane of scattering is $\sin^2\phi$. Factor (3) may now be obtained by multiplying the probability that the light is polarized in the plane of the scattering by the factor $k/k_0+k_0/k-2\sin^2\theta$, and adding to this product the product of the probability that the light is polarized at right angles to the plane of scattering by its corresponding factor $k/k_0+k_0/k$. The fact that the relative intensity for an arbitrary angle of polarization can be computed in terms of the probability of polarization in the plane and at right angles to the plane of scattering shows that the scattering of a quantum by an electron produces a partial analysis in terms of plane-polarized light in the plane of scattering, and at right angles to this plane. This result holds if and only if the polarizations of the quanta are resolved in this particular way.

We now suppose that the quantum with momentum $\mathbf{k}_0$ is scattered through an angle θ_1 into a photon of momentum $\mathbf{k}_1$. The *a priori* probability that this light quantum had its plane of polarization in the plane of scattering is one-half, and that it had its plane of polarization at right angles to the plane of scattering is one-half. Thus, as a consequence of the scattering the *a posteriori* probability that this quantum had its plane of polarization in the plane of the scattering is

$$\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}-2\sin^2\theta_1\right)\Big/2\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}-\sin^2\theta_1\right),$$

and the *a posteriori* probability that its plane of polarization was perpendicular to the plane of scattering is

$$\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}\right)\Big/2\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}-\sin^2\theta_1\right).$$

According to the wave functional (2) this means that the probability that the second quantum, whose momentum is $-\mathbf{k}_0$, has its plane of polarization in the plane of scattering of the first light quantum is

$$\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}\right)\Big/2\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}-\sin^2\theta_1\right),$$

and the probability that the second light quantum's plane of polarization is perpendicular to the plane of scattering of the first light quantum is

$$\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}-2\sin^2\theta_1\right)\Big/2\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}-\sin^2\theta_1\right).$$

If we now apply the above probabilities to the calculation of the scattering of the second quantum by an electron we find that the normalized probability distribution, dP, for the two light quanta, assuming that both the quanta are scattered, is given by

$$dP=\frac{k_1^2k_2^2(\gamma_1\gamma_2-\gamma_1\sin^2\theta_2-\gamma_2\sin^2\theta_1+2\sin^2\theta_1\sin^2\theta_2\sin^2\phi)d\Omega_1d\Omega_2}{4\pi^2k_0^4\left(\frac{40}{9}-3\ln 3\right)^2},\tag{4}$$

$$\gamma_1=\frac{k_1}{k_0}+\frac{k_0}{k_1},\quad \gamma_2=\frac{k_2}{k_0}+\frac{k_0}{k_2}.$$

In this equation $d\Omega_1$ and $d\Omega_2$ are differential solid angles for the scattered first and second light quanta, k_1 and k_2 are the magnitudes of their momenta, θ_1 and θ_2 are their scattering angles, and φ is the angle between the two planes of scattering. Also, for the case of practical interest, in which the kinetic energies of the positron and electron are small, the two quanta have an energy of mc^2 each and the Compton formula gives

$$k_1=\frac{k_0}{2-\cos\theta_1},\quad k_2=\frac{k_0}{2-\cos\theta_2}.\tag{5}$$

Equations (5) were used in the normalization of (4). When $\theta_1=\theta_2=\theta$, Eq. (4) leads to the ratio given in Eq. (1).

3. PERTURBATION THEORY METHOD

It is of interest to verify the above conclusion by a direct application of time-dependent perturbation theory. The process involved is a four-quanta process, corresponding to absorption and emission of quanta by the two electrons involved in the scattering. However, since the scattering processes are independent (except for the connection between polarizations of the initial quanta) the problem reduces to consideration of a pair of two-quanta processes.

In the standard time-dependent perturbation theory[5] the perturbation H' is replaced by $\lambda H'$, and the wave function ψ is expressed as a power series in λ.

$$\psi=\psi^{(0)}+\lambda\psi^{(1)}+\lambda^2\psi^{(2)}+\cdots,\tag{6}$$

where $\psi^{(0)}$ is the wave function of the unperturbed state. The $\psi^{(s)}$ are then expanded in terms of the eigenfunctions of the unperturbed time-dependent wave equation

$$\psi^{(s)}=\sum_n a_n^{(s)}u_ne^{-iE_nt/\hbar}\quad(s=0,1,2,\cdots).$$

Substitution into the perturbed wave equation, equating coefficients of equal powers of λ, and setting $\lambda=1$, yields the equations

$$\dot{a}_m^{(0)}=0,\quad \dot{a}_m^{(s+1)}=\frac{1}{i\hbar}\sum_n H'_{mn}a_n^{(s)}e^{i\omega_{mn}t},$$

where

$$\omega_{mn}=(E_m-E_n)/h;\quad H'_{mn}=\int u_m^*H'u_nd\mathbf{r}.$$

For a two-quanta process, the solution of these

[5] See, for example, W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1944), second edition, p. 87. We shall follow essentially Heitler's notation in the remainder of this section. In particular, the wave functions of the plane waves involved in the determination of matrix elements are normalized to unit volume, and the momenta are expressed in energy units (i.e., the quantity $c\times$ momentum is called momentum).

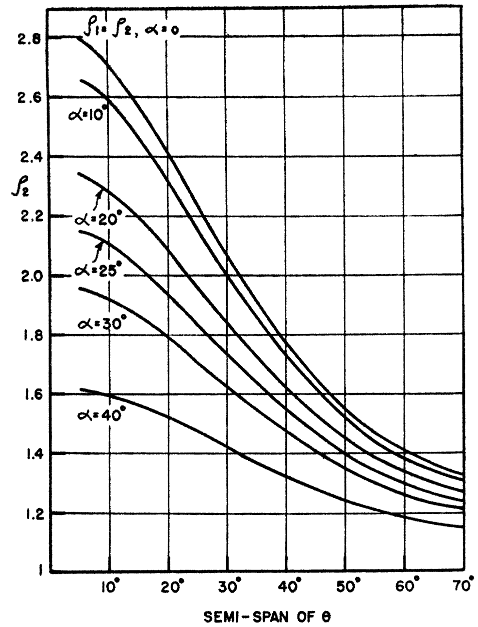

FIG. 4. Asymmetry in coincidence counting rates, $\rho_2 = P_{\text{orthogonal}}/P_{\text{coplanar}}$, for finite geometry, as function of half-span in θ for various half-spans in azimuth, α. Curve for $\alpha=0$ also represents asymmetry ratio $\rho_1 = \Delta P_{\phi=\pi/2}/\Delta P_{\phi=0}$.

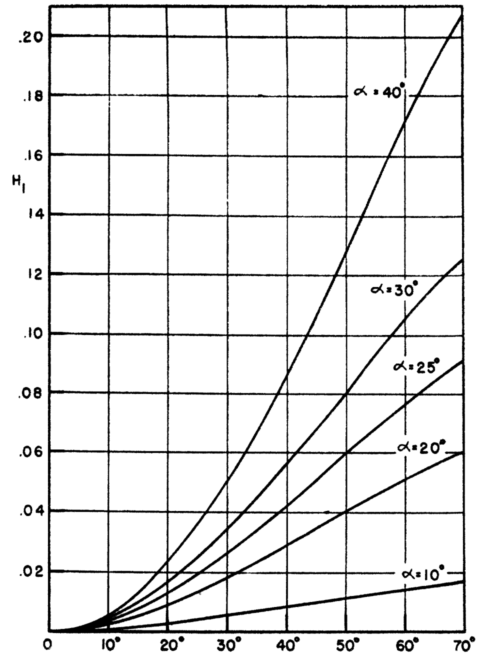

FIG. 5. Geometric efficiency, H_1, of coincidences in orthogonal position, as function of half-span in θ for various half-spans in azimuth, α.

equations subject to the initial conditions

$$a_m^{(0)} = \delta_{mP},$$

is

$$a_F^{(2)}(t, P) = \frac{1}{\hbar^2} \sum_n H'_{Fn} H'_{nP} \frac{1}{\omega_{nP}} \times \left\{ \frac{e^{i(\omega_{nP}+\omega_{Fn})t}-1}{\omega_{nP}+\omega_{Fn}} - \frac{e^{i\omega_{Fn}t}-1}{\omega_{Fn}} \right\}. \quad (7)$$

Here the wave functional of the unperturbed state is (deleting for convenience the functions representing the electrons)

$$\frac{1}{\sqrt{2}}\{\psi_A(1)\psi_B(2)+\psi_C(1)\psi_D(2)\}, \quad (8)$$

where

$$\begin{aligned} \psi_A(1) &= \psi(\cdots 1_{\mathbf{k}_0 1}, 0_{\mathbf{k}_0 2}, \cdots), \\ \psi_B(2) &= \psi(\cdots 0_{-\mathbf{k}_0 1}, 1_{-\mathbf{k}_0 2}, \cdots), \\ \psi_C(1) &= \psi(\cdots 0_{\mathbf{k}_0 1}, 1_{\mathbf{k}_0 2}, \cdots), \\ \psi_D(2) &= \psi(\cdots 1_{-\mathbf{k}_0 1}, 0_{-\mathbf{k}_0 2}, \cdots). \end{aligned} \quad (9)$$

The Hamiltonian of the unperturbed system and the perturbation Hamiltonian may both be written as sums of Hamiltonians for the two sub-systems. If we replace $H'(1)+H'(2)$ by $\lambda_1 H'(1) + \lambda_2 H'(2)$, the wave functional for the complete system may be written in the form

$$\psi = \frac{1}{\sqrt{2}}\{\psi_a(1)\psi_b(2)+\psi_c(1)\psi_d(2)\}, \quad (10)$$

where $\psi_a(1)$, $\psi_c(1)$ may separately be expanded as power series in λ_1 of the form (6), and $\psi_b(2)$, $\psi_d(2)$ may be expanded as power series in λ_2; and

$$\begin{aligned} \psi_a^{(0)}(1) &= \psi_A(1), & \psi_b^{(0)}(2) &= \psi_B(2), \\ \psi_c^{(0)}(1) &= \psi_C(1), & \psi_d^{(0)}(2) &= \psi_D(2). \end{aligned} \quad (11)$$

The usual reduction procedure then replaces (7) by

$$a_F^{(2,2)}(t_1, t_2) = a_F^{(2)}(t_1, A) a_F^{(2)}(t_2, B) + a_F^{(2)}(t_1, C) a_F^{(2)}(t_2, D). \quad (12)$$

The transition probability for the joint scattering process (if one observes one sub-system at time t_1 and the other at time t_2) is $|a_F^{(2,2)}|^2$. It may be reduced in the usual manner, by assuming for each sub-system that in the neighborhood of the final state there are a large number of states with the same physical properties and that $\rho_E dE$ represents the number of these states with energy between E and $E+dE$. The probability dP' of finding the system with the photons in any one of the appropriate states is

$$dP' = \int_{\Delta E_1}\int_{\Delta E_2} |a_F^{(2,2)}|^2 \rho_{E1}(1)\rho_{E2}(2)dE_1dE_2$$

$$= \frac{4\pi^2 t_1 t_2}{h^2}\rho_{E1}\rho_{E2}\cdot\tfrac{1}{2}|R(1)S(2)+S(1)R(2)|^2$$

$$= \frac{2\pi^2 t_1 t_2}{h^2}\rho_{E1}\rho_{E2}\{|R(1)|^2|S(2)|^2 + |S(1)|^2|R(2)|^2 + R(1)S^*(1)R^*(2)S(2) + R^*(1)S(1)R(2)S^*(2)\}, \quad (13)$$

where

$$R=\sum_{n'}\frac{H'_{An'}H'_{n'F}}{E_0-E_{n'}}, \quad S=\sum_{n'}\frac{H'_{Cn'}H'_{n'F}}{E_0-E_{n'}},$$

$$E_0=\tfrac{1}{2}E_A=\mu+k_0, \quad E_i=\mu+k_0-k_i \; (i=1,\,2), \quad (14)$$

and the arguments refer to the two photons. The electronic mass μ and the momenta k_0, k_1, and k_2 are expressed in energy units in accordance with Heitler's notation.

It is of interest to note here that for small times t_1 and t_2 (but t_1 and $t_2 \gg h/E_0$) the transition probability dP' is proportional to the product t_1t_2. This is to be expected, since we deal here with a joint probability of two scattering processes. It is therefore inappropriate to define a transition probability per unit time. Similarly, it is not appropriate to call $dP'/c^2t_1t_2$, which has the dimensions of (length),[4] a "differential cross section" for the scattering. However, the difficulty is of no consequence in the problem we are treating, since we are concerned only with the relative probability of different angular relations of the scattered photons.

A result equivalent to (12) and (13) could also be obtained by regarding the process as a four-quanta process, corresponding to absorption and emission of quanta by the two electrons involved in the scattering. For any sequence of absorptions and emissions there are five other sequences involving the same energy differences, namely those in which the same four absorption and emission processes take place in a different order, retaining, however, the order of absorption and emission of quanta by each electron. If in summing over intermediate states we sum first over such permutations, the resultant terms involving exponentials in t factor into a product of two terms of the form given in Eq. (7). The coefficient $a_F^{(4)}(t)$ is thus reduced to $a_F^{(2,2)}$ as given in Eq. (12), with $t_1=t_2=t$. The transition probability dP' is then proportional to t^2.

The matrix elements R and S may be reduced by the procedure used in treating the scattering of one photon, and yield

$$R(i)=\frac{2\pi e^2 h^2 c^2}{(k_0k_i)^{\frac{1}{2}}}\sum\left\{\frac{(u_0{}^*\alpha_{01}u')(u'{}^*\alpha u)}{\mu+k_0-E'} + \frac{(u_0{}^*\alpha u'')(u''{}^*\alpha_{01}u)}{\mu-k_i-E''}\right\},$$

$$S(i)=\frac{2\pi e^2 h^2 c^2}{(k_0k_i)^{\frac{1}{2}}}\sum\left\{\frac{(u_0{}^*\alpha_{02}u')(u'{}^*\alpha u)}{\mu+k_0-E'} + \frac{(u_0{}^*\alpha u'')(u''{}^*\alpha_{02}u)}{\mu-k_i-E''}\right\}, \quad (15)$$

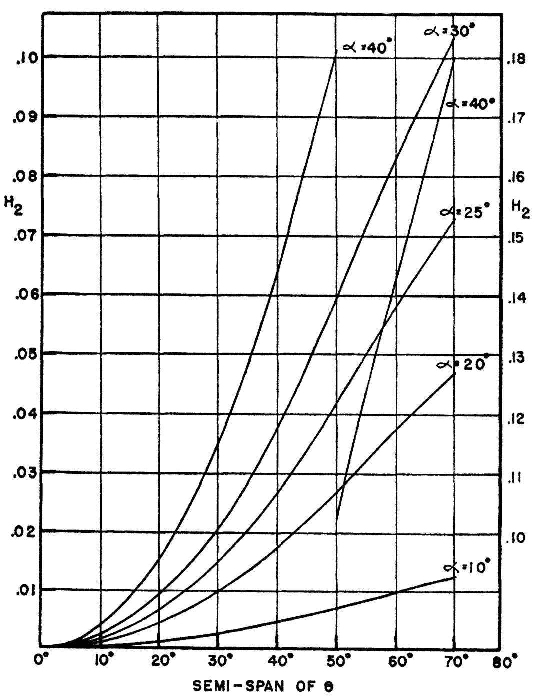

FIG. 6. Geometric efficiency, H_2, of coincidences in coplanar position, as function of half-span in θ for various half-spans in azimuth, α. Ordinate scale at right for curve $\alpha=40°$, $\theta>50°$.

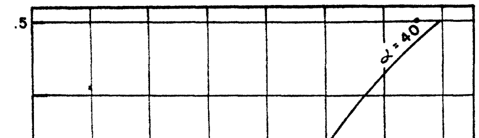

FIG. 7. Relative geometric efficiency for coincidences following single counts, E_1, in orthogonal position. Dashed curves connect points $\rho_2=$constant.

where i takes the values 1 and 2. The symbols α_{01} and α_{02} represent the components of the Dirac vector $\boldsymbol{\alpha}$ in the directions $\boldsymbol{\varepsilon}_{\mathbf{k}01}$ and $\boldsymbol{\varepsilon}_{\mathbf{k}02}$, respectively. The densities ρ_{E_i} are given by the usual expression

$$\rho_{E_i}=\frac{E_ik_i}{\mu k_0}\frac{k_i^2d\Omega_i}{(2\pi\hbar c)^3}\quad(i=1,2). \tag{16}$$

The terms in Eq. (13) may be evaluated in the usual way, by summing over the spin directions of the scattering electrons, averaging over their spin directions in the initial state, and evaluating spurs. For instance, the Klein-Nishina formula derivation gives

$$|R(1)|^2=\frac{\pi e^2\hbar^2c^2}{2\mu E_0k_0k_1}\times\left(\frac{k_1}{k_0}+\frac{k_0}{k_1}-2+4\cos^2\Theta_{1,01}\right), \tag{17}$$

where $\Theta_{1,01}$ is the angle between the polarization direction of $\mathbf{k}_1$ and $\boldsymbol{\varepsilon}_{\mathbf{k}01}$. Similar expressions hold for $|R(2)|^2$, $|S(1)|^2$, and $|S(2)|^2$. We must also sum over the states of polarization of the scattered quanta, since we are interested in all the photons of momenta $\mathbf{k}_1$ and $\mathbf{k}_2$.

The evaluation outlined in the preceding paragraph is simplified considerably by the proper choice of two mutually perpendicular directions for the polarization of one of the scattered photons. For instance, if the polarization directions of $\mathbf{k}_1$ are taken to be perpendicular to the scattering plane ($\boldsymbol{\varepsilon}_1=\boldsymbol{\varepsilon}_{\mathbf{k}02}$) and in the scattering plane ($\boldsymbol{\varepsilon}_1=\boldsymbol{\varepsilon}_{\mathbf{k}01}\cos\theta_1-\boldsymbol{\varepsilon}_{\mathbf{k}03}\sin\theta_1$) the cross-product terms in Eq. (13) vanish. For these terms reduce in the usual way to an evaluation of spurs of which a typical example is

$$\text{spur }\alpha_{02}K'\alpha_1(E-\beta\mu-\boldsymbol{\alpha}\cdot\mathbf{p}_1)\alpha_1K'\alpha_{01}(1+\beta), \tag{18}$$

where

$$K'=\mu(1+\beta)+k_0+\boldsymbol{\alpha}\cdot\mathbf{k}_0,$$
$$\mathbf{p}_1=\mathbf{k}_1-\mathbf{k}_0.$$

The given choices of polarization directions give $\alpha_1=\alpha_{02}$, and $\alpha_1=\alpha_{01}\cos\theta_1-\alpha_{03}\sin\theta_1$, and in either case it is evident that the spur (18) contains an odd number of α_{02} terms and hence vanishes. We may now choose also any two mutually perpendicular polarization directions for the scattered quantum $\mathbf{k}_2$. A convenient pair are perpendicular to the scattering plane of $-\mathbf{k}_0$ and in its scattering plane.

Combining Eqs. (13), (16), and (17), and summing over the polarization directions of $\mathbf{k}_1$ and $\mathbf{k}_2$, we obtain

$$\frac{dP'}{c^2t_1t_2}=\frac{1}{16}r_0^4d\Omega_1d\Omega_2\frac{k_1^2}{k_0^2}\frac{k_2^2}{k_0^2}(\gamma_1\gamma_2-\gamma_1\sin^2\theta_2-\gamma_2\sin^2\theta_1+2\sin^2\theta_1\sin^2\theta_2\sin^2\varphi), \tag{19}$$

where

$$r_0=e^2/\mu=e^2/mc^2,$$
$$\gamma_i=(k_i/k_0)+(k_0/k_i),$$
$$k_i=k_0/(2-\cos\theta_i)\quad(i=1,2). \tag{20}$$

Since we are interested only in the correlation of the scattering quanta when both original quanta are known to be scattered, Eq. (19) should be normalized so that the integral over all directions of scattering of both quanta gives unit probability. This yields the result given in Eq. (4).

In Eq. (13), it is evident that the sum of $|R(1)|^2$ over the polarization states of $\mathbf{k}_1$ will give the same value no matter which two perpendicular directions are chosen for $\boldsymbol{\varepsilon}_1$ (so long as they are perpendicular to $\mathbf{k}_1$, of course). Similar statements hold for $|S(1)|^2$, $|R(2)|^2$, and $|S(2)|^2$. The cross-product terms must therefore vanish when summed over the polarization states of the final quanta. (However, it is only for the choices of $\boldsymbol{\varepsilon}_1$ mentioned above that they vanish before the sum is taken.) This reduction of the expression in brackets in Eq. (13) to $\{|R(1)|^2|S(2)|^2+|S(1)|^2|R(2)|^2\}$ is equivalent to the conclusion in Section 2 that the scattering of one quantum by an electron produces a partial analysis in terms of plane-polarized light in the plane of scattering and at right angles to this plane, and hence gives a partial analysis also of the second quantum emitted in the annihilation process.

4. FINITE GEOMETRY

If an experimental verification of the expected asymmetry in coincidence counting rates is attempted, it will be found that a close approach to ideal geometry is impractical. It is, therefore, desirable to extend the calculations to the case where the gamma-ray counters subtend finite angles.

We set $\varphi=\varphi_2-\varphi_1$, with φ_1 and φ_2 the azimuths of an element of counters 1 and 2, respectively, relative to the axis of counter 1. In integrating Eq. (4) over finite ranges in θ_1, θ_2, and φ_1, φ_2 it will be sufficient to consider the arrangement in which the two counters subtend equal angles.

First we perform the integration over θ. With $d\Omega_i=\sin\theta_i d\theta_i d\varphi_i$, $(i=1,2)$, $1/q=4\pi^2(40/9-3\ln 3)^2$, and the abbreviations $x=2-\cos\theta$, $\gamma=x+1/x$ and

$$J=\int \gamma(k/k_0)^2\sin\theta d\theta=\ln x-1/2x^2,$$

$$J'=\int (k/k_0)^2\sin^3\theta d\theta=-x+4\ln x+3/x, \tag{21}$$

we obtain from Eq. (4)

$$\Delta P=d\varphi_1 d\varphi_2\int dP d\theta_1 d\theta_2$$

$$=q(J^2-2JJ'-2J'^2\sin^2\varphi)d\varphi_1 d\varphi_2, \tag{22}$$

and for the asymmetry for finite range in θ but infinitesimal range in φ

$$\rho_1=\frac{\Delta P_{\varphi=\pi/2}}{\Delta P_{\varphi=0}}=1+\frac{1}{\frac{1}{2}(J/J')^2-(J/J')}. \tag{23}$$

The integrals J and J' were evaluated for intervals symmetrical about $\theta=82°$ which is the approximate location of the maximum of ρ (see Fig. 2). ρ_1 as a function of the half-span of such intervals is included in Fig. 4 ($\alpha=0$).

The calculations are then extended to finite ranges in φ_1 and φ_2. We obtain P_1 by integrating Eq. (22) over the intervals $-\alpha<\varphi_1<\alpha$ and $\pi/2-\alpha<\varphi_2<\pi/2+\alpha$, and P_2 for the intervals $-\alpha<\varphi_{1,2}<\alpha$. With

$$\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\sin^2(\varphi_2-\varphi_1)d\varphi_1 d\varphi_2=2\alpha^2-\tfrac{1}{2}\sin^2 2\alpha=u,$$

$$\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\cos^2(\varphi_2-\varphi_1)d\varphi_1 d\varphi_2=2\alpha^2+\tfrac{1}{2}\sin^2 2\alpha=w,$$

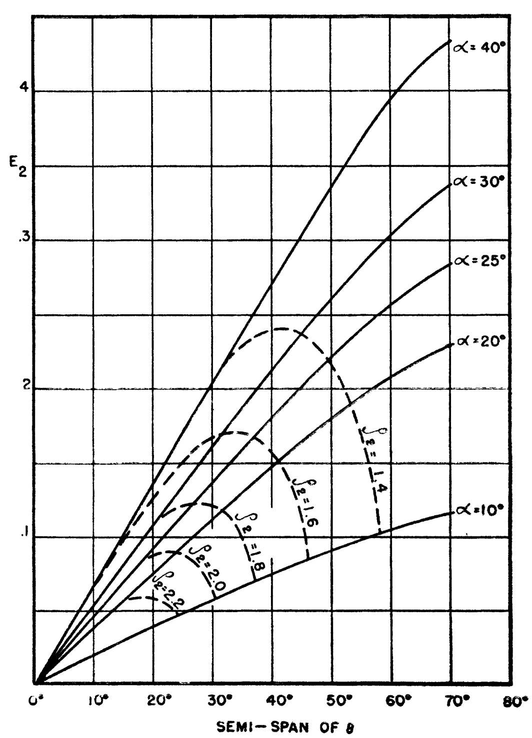

FIG. 8. Relative geometric efficiency for coincidences following single counts, E_2, in coplanar position. Dashed curves connect points ρ_2=constant.

and $z=u/w$, the asymmetry $\rho_2=P_1/P_2$ can be written as

$$\rho_2=(z+\rho_1)/(1+z\rho_1). \tag{24}$$

Curves of ρ_2 as a function of the above-mentioned half-spans in θ and for various values of α are shown in Fig. 4.

In setting up the experiment it is also required to know the efficiencies. We shall here give the geometric part of the efficiencies, which are defined as the number of pairs of scattered quanta reaching both counters divided by the number of all scattered pairs. To obtain over-all efficiencies, the geometric factors must, of course, be multiplied by the efficiencies of the counters[6] and the fractions of the incident intensities scattered by the targets.

Using, as before, subscripts 1 and 2 for the orthogonal and coplanar positions, we derive from Eq. (22) the efficiencies η_i for "semifinite" geometry

$$\begin{aligned}\eta_2&=\frac{J^2-2JJ'}{J_0{}^2-2J_0J_0'}=\frac{J'^2}{J_0'^2}\left(\frac{\rho_{10}-1}{\rho_1-1}\right),\\ \eta_1&=\frac{\rho_1}{\rho_{10}}\eta_2.\end{aligned} \tag{25}$$

Here, the η's have been normalized to unity for the range $0<\theta<\pi$, i.e., $\eta_i=\Delta P_i/\Delta P_0$; the subscript 0 referring to this range.

Integration of Eq. (22) leads to the following expressions for the geometric efficiencies H_i for finite longitudinal and azimuthal apertures: (In the following equations we assume two detectors on each side, relatively displaced by 180° in azimuth and covering the same range in θ.)

$$\begin{aligned}H_2&=\frac{2}{\pi^2}\left(\frac{u\eta_1}{1+1/\rho_{10}}+\frac{w\eta_2}{1+\rho_{10}}\right),\\ H_1&=\rho_2H_2=\frac{2}{\pi^2}\left(\frac{w\eta_1}{1+1/\rho_{10}}+\frac{u\eta_2}{1+\rho_{10}}\right).\end{aligned} \tag{26}$$

The functions H_1 and H_2 are shown, for the same intervals as before, in Figs. 5 and 6.

Also needed in designing the experiments are the relative geometric efficiencies E_i, i.e., the probabilities of registering a coincidence once a single count has been observed. These are obtained in terms of H_i and the efficiency D for single counts

$$\begin{aligned}E_1&=H_1/D,\\ E_2&=H_2/D,\end{aligned} \tag{27}$$

with

$$D=\frac{2\alpha}{\pi}\left(\frac{J-J'}{J_0-J_0'}\right). \tag{28}$$

Equation (28) is, of course, directly obtained by integration of the Klein-Nishina formula with respect to θ (after summing over polarization directions of the incident and scattered quanta). The factor of 2 enters since there are two single scattering processes for each scattered pair.

The functions E_i are plotted in Figs. 7 and 8. The dotted curves in these plots connect the points $\rho_2=$constant. It is seen that for a given value of ρ_2 best efficiencies are obtained for approximately square apertures.

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[6] See, for example, Bradt, Gugelot, Huber, Medicus, Preiswerk, and Scherrer, Helv. Phys. Acta. 19, 77 (1946). Note that our calculations do not take into account the energy dependence of the counter sensitivity which results in a dependence of detector sensitivity on θ.