

Disintegration of Slow Mesotrons

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The disintegration of mesotrons at the end of their range was investigated by means of an improved arrangement of the type already described by the author. The absorption of a mesotron by a block of aluminum or iron is recorded by a system of coincidence and anticoincidence counters. Another system of counters and circuits registers the delayed emission of a particle, which is interpreted as the disintegration electron associated with the absorbed mesotron. The present apparatus enables one to determine the time distribution of the emitted particles and hence

the mean life of the decay process, independently of the effects produced by the scattering of mesotrons. The mean life is found to be 1.5 ± 0.3 microseconds, in substantial agreement with the value deduced from the atmospheric absorption effect. The absolute number of disintegration electrons per absorbed mesotron has also been determined (for an Al absorber) and found to be about one-half. This result suggests that, in agreement with theoretical predictions, positive mesotrons undergo spontaneous decay, while the negative ones react with nuclear particles.

1. INTRODUCTION

THE anomalous absorption of the hard component of cosmic rays in the atmosphere has received a satisfactory explanation through the assumption of the mesotron decay. Recent and more accurate experiments^{1,2} have confirmed the hypothesis, which was at first based on rather doubtful experimental material, and given a fairly accurate evaluation of the proper lifetime of the free mesotron.

Nevertheless, there remains much to be learned about the process of mesotron disintegration. Since the mesotron is usually identified with the particle which, in the theory of Yukawa,³ is responsible for the nuclear forces, it is assumed to transform, by a process of β -decay, into an electron and a neutrino. The electrons should, on the average, take up half of the energy of the disintegrating mesotrons. Whether the intensity of the electron component in the atmosphere is sufficiently high to agree with the measured lifetime of the mesotron, is a much discussed and still unsettled point.^{4,5}

Even assuming that the process of disintegration of free mesotrons were completely known, there would remain the important point of determining what happens to the mesotrons

that have been brought to the end of their range by ionization energy losses. According to the Yukawa theory, the mesotron possesses strong interactions with the heavy particles. The most important process for a slow (negative) mesotron would be an analog of the photoelectric effect,⁶ by which the mesotron would disappear transferring its total energy to a bound proton which would be transformed into a neutron. A similar process could take place between a positive mesotron and a neutron.

Recent calculations by Tomonaga and Araki⁷ seem to indicate that, owing to the Coulomb field of the nucleus, which attracts the negative mesotrons and repels the positive ones, the above-mentioned process should be much less probable than spontaneous decay for positive mesotrons. For negative mesotrons, on the contrary, nuclear interactions should be so probable that spontaneous decay of mesotrons at rest in dense materials should not take place.

Prior to the author's work, the only observation of mesotrons decaying at the end of their range were two cloud-chamber tracks photographed by Williams and Roberts.⁸

Preliminary reports⁹ announced the observation of disintegration electrons by means of a counter arrangement and a rough measurement of the mean life. The present paper contains an

¹ B. Rossi and D. B. Hall, *Phys. Rev.* **59**, 223 (1941).

² W. M. Nielsen, C. M. Ryerson, L. W. Nordheim, and K. Z. Morgan, *Phys. Rev.* **59**, 547 (1941).

³ H. Yukawa, *Proc. Phys. Math. Soc. Japan* **20**, 319 (1938).

⁴ G. Bernardini, B. N. Cacciapuoti, B. Ferretti, O. Piccioni, and G. C. Wick, *Phys. Rev.* **58**, 1017 (1940).

⁵ L. W. Nordheim, *Phys. Rev.* **59**, 554 (1941).

⁶ H. Euler and H. Heisenberg, *Ergeb. d. exakt. Naturwiss.* **17**, 1 (1938).

⁷ S. Tomonaga and G. Araki, *Phys. Rev.* **58**, 90 (1940).

⁸ E. J. Williams and G. E. Roberts, *Nature* **145**, 102 (1940).

⁹ F. Rasetti, *Phys. Rev.* **59**, 706, 613 (1941).

account of more complete and precise measurements, whose aim was not only to determine directly the mean life of the decay process, but also to attempt to settle the important point whether all slow mesotrons, or only half of them, undergo spontaneous disintegration.

2. EXPERIMENTAL PROCEDURE

Counter arrangement

The principle adopted consists of selecting, by means of a coincidence-anticoincidence system of counters, the events in which a mesotron is stopped in an absorbing block (of iron or aluminum) and observing the delayed emission of an ionizing particle from the absorber. The delayed particle, detected by means of a suitable system of counters and circuits, is assumed to be the disintegration electron of the absorbed mesotron.

The general scheme of the counter set-up and of the recording circuits is illustrated in Fig. 1. Counters designated by the same letter are connected in parallel. Cross-marked counters are employed as anticoincidence counters or anticounters. The circles represent the effective cross sections of the counters.

The fourfold coincidence system ($ABCD$) defines a beam of mesotrons, the soft component of cosmic rays being filtered out by 15 cm of lead placed above and between the counters. Counters A , B , C , and D have an effective length of 22 cm. The anticounters F (effective length 52 cm) cover the whole solid angle subtended by the system ($ABCD$), so that anticoincidences ($ABCD-F$), when no absorber is present between D and F , are mostly due to lack of efficiency of the counters F . This number is found to be 1.5 percent of the number of coincidences ($ABCD$).

The absorber has a cross section of $2.5 \times 10 \text{ cm}^2$ and a length of 40 cm. About 75 percent of the mesotrons defined by the coincidences ($ABCD$) traverse the absorber. The two anticounters in position G were added, in parallel with the others, to discriminate against mesotrons associated with showers.

The six counters E are placed in such a way that a particle emitted from the absorber has a fairly large probability of being detected by one of them. They have an effective length of 37 cm,

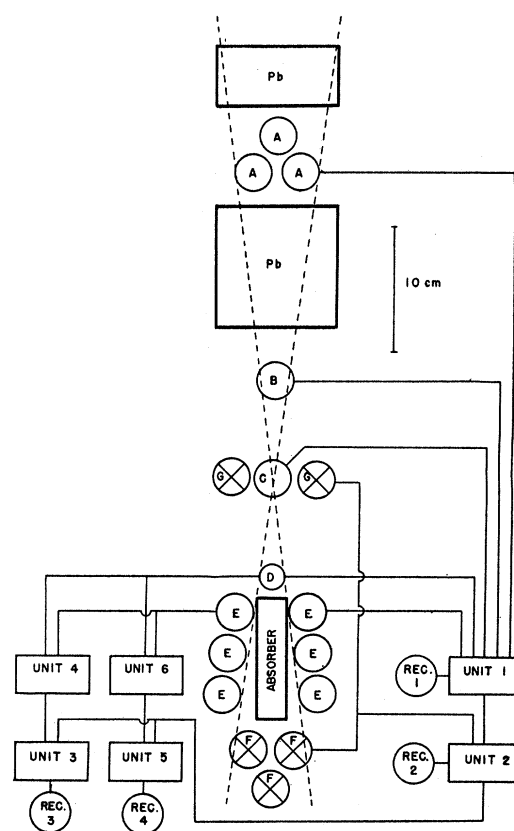


FIG. 1. Arrangement of counters, illustrating connections to amplifier units.

and lie wholly outside of the solid angle defined by the system ($ABCD$), as is clearly indicated in Fig. 1.

The inside diameter of all counters is 29 mm, except for counter D which has a diameter of 20 mm. The counters have a brass cathode and a 4-mil tungsten wire as anode. The electrodes are sealed in a glass tube filled with a mixture of commercial argon (5 cm Hg) and ethyl alcohol vapor (1 cm Hg). Extreme care was employed in selecting the counters, especially the critical ones of groups D and E , which are connected to the high resolving power amplifiers. The pulses were carefully studied on a cathode-ray oscilloscope, and no counter was accepted unless the pulses were all of equal size and of satisfactory shape. Most counters were found to answer the requirements, provided that the leak resistance was not lower than 10^8 ohms, and the coupling capacity not larger than 10 or 20 μmf . Such values of these

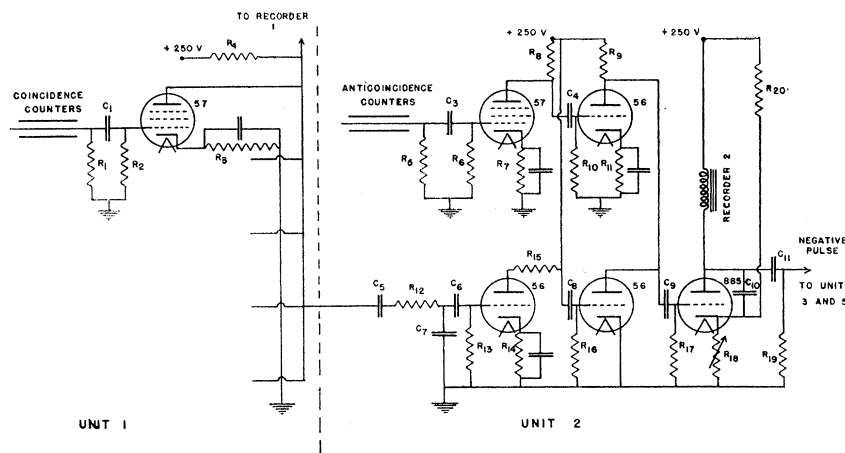


FIG. 2. Fivefold coincidence circuit (unit 1) and anticoincidence circuit (unit 2). Only one of the five Rossi tubes is shown. $R_1 = R_5 = 10^8$; $R_2 = R_6 = R_8 = 500,000$; $R_3 = 5000$; $R_4 = 1$ Meg; $R_7 = 3000$; $R_9 = 7500$; $R_{10} = R_{13} = R_{16} = 2$ Meg; $R_{11} = R_{19} = 50,000$; $R_{12} = 200,000$; $R_{14} = 30,000$; $R_{15} = 15,000$; $R_{17} = 300,000$; $R_{18} = 10,000$ adjustable; $R_{20} = 25,000$; $C_1 = 0.00001$; $C_3 = 0.00005$; $C_4 = 0.03$; $C_5 = 0.0001$; $C_6 = 0.001$; $C_7 = 0.00003$; $C_8 = C_9 = 0.001$; $C_{10} = 0.1$; $C_{11} = 0.0001$. Resistance in ohms, capacity in μf .

constants were actually employed (see Figs. 2 and 3).

The essential check of the counters, however, consisted in showing that they were practically 100 percent efficient in recording double coincidences when the resolving time of the circuit was as low as 5×10^{-7} sec. This check will be discussed in a later section.

The counters had a threshold voltage of about 900 volts and were operated at 1050 volts through a pentode-stabilized circuit. No replacement of any of the counters and no readjustment of the voltage were required during a series of measurements which extended over a period of five months. The size and shape of the pulses of each counter were checked on the cathode-ray oscilloscope at least once a week.

Circuits

The general scheme of the recording circuits has already been described,⁹ except for the addition of the units 5 and 6 (Fig. 1) and may be briefly explained as follows. Counters *A*, *B*, *C*, *D*, and *E* are connected to a fivefold coincidence set (unit 1). The pulse from unit 1 is passed onto the anticoincidence unit 2, so that recorder 2 is operated only when a coincident discharge (*ABCDE*) is not associated with a discharge of *F* (this process will be designated by (*ABCDE* - *F*)). For instance, a mesotron which discharges counters *A*, *B*, *C* and *D*, is stopped by the absorber, and emits a disintegration electron which trips one of the counters *E*, will be registered by

recorder 2. A scattered mesotron may produce the same effects.

The negative pulse from the plate of the thyatron which operates recorder 2 is passed onto both the double coincidence units 3 and 5. Each one of these operates a recorder (recorders 3 and 4) only when a coincident pulse (within about 10^{-4} sec.) is transmitted to it by, respectively, unit 4 or 6. These are two double coincidence sets of short resolving time, both connected to counters *D* and *E*.

Thus, recorder 2 registers a process (*ABCDE* - *F*), recorders 3 and 4 each a process which we may designate by $\{(ABCDE - F)(DE)\}$. Were all coincidences between counters produced by the same particle, or by particles associated in a practically instantaneous process (such as a shower), the numbers of events n_2 , n_3 , and n_4 registered, respectively, by recorders 2, 3, and 4 would be exactly equal. However, delayed processes such as the disintegration of a mesotron may give a fivefold coincidence (*ABCDE*) on unit 1 which is not recorded as a double coincidence (*DE*) on unit 6, or on both units 4 and 6, on account of their shorter resolving time.

The essentials of units 1, 2, 4, and 6 are illustrated in Figs. 2 and 3. Unit 1 is a conventional Rossi fivefold coincidence set. Its output pulse may be fed to a thyatron operating recorder 1, besides being passed onto the anticoincidence unit 2. The anticoincidence circuit does not embody any novel features, but since it proved to be simple and 100 percent efficient, its diagram

is given in Fig. 2. The capacity-resistance circuit $R_{12}-C_7$ delays the positive pulse from unit 1. The pulse from the anticounters is longer and after two stages of amplification has a square form, amply covering the time interval occupied by the coincidence pulse.

The short resolving time units 4 and 6 are three-stage double coincidence amplifiers, the Rossi tubes constituting the third stage, and comprise a type 885 thyatron which transmits a negative pulse to, respectively, unit 3 or 5 (Fig. 3). All coupling resistances, of course, are low, and capacities small, to insure selective amplification of the high frequency components of the pulse and make the resolving time short. The two units differ only in the value of the coupling capacity C_2 and in the bias of the thyatron cathode, which is determined by the adjustable resistance R_{13} .

All leads from the counters to the amplifiers are provided with low capacity shielding.

The two double coincidence units 3 and 5 are conventional Rossi circuits and do not need description.

Measurement of resolving time

The procedure employed was the usual one of counting the number n of random coincidences when the counters were operated at high counting rates, N_1 and N_2 , measured by means of a vacuum tube scaling circuit. At counting rates of several thousands per minute (as are necessary if random coincidences are to be recorded in sufficient number by a circuit having a resolving time of the order of one microsecond), a fraction of the pulses has a size smaller than normal, due to the slow recovery of the voltage across the counter after a discharge. This led to an investigation of the dependence of resolving time upon counting rate and pulse size.

All other counters being left in their normal position, counter D was displaced at a considerable distance from group E , without in the least changing the length of the leads, stray capacities, or other factors which may affect the resolving time. Counter D was also shielded by 20 cm lead. Under such conditions, the rate of systematic coincidences (DE) was only 2.0 ± 0.3 per hour. Counters D and E were then irradiated with weak radioactive sources, and the number of random

coincidences (DE) was simultaneously registered on the three units 1, 4 and 6. For this measurement, the circuits were employed in exactly the same way as in the actual experiments, except for unit 1, which is here used only as a double coincidence circuit. However, the exact knowledge of its resolving time is not required for the evaluation of the final experiments.

Table I contains the results on the dependence of resolving time upon counter voltage (which affects the pulse size) and upon counting rate. The resolving time (respectively t_2 , t_3 and, t_4 for units 1, 4, and 6) was calculated from the usual formula $t = n/2N_1N_2$.

The result is that the resolving time does not appreciably change within the range of voltage and counting rate investigated. Since the pulses at high counting rates ($2N_1N_2 = 4900 \text{ sec.}^{-2}$) have the same average size at 1065 volts as they have at background rate and 1035 volts, it seems reasonable to assume that the resolving times measured are valid for the conditions under which the counters were actually employed in the experiments.

The measurements reported in Table I were performed before the beginning of the runs. At the end of the run with an iron absorber, another determination (at 1050 volts) gave: $t_2 = 14.7 \pm 0.3$; $t_3 = 2.08 \pm 0.1$; $t_4 = 0.85 \pm 0.07$. At the end of the run with an aluminum absorber the values found were: $t_2 = 14.0 \pm 0.2$; $t_3 = 1.92 \pm 0.1$; $t_4 = 0.66 \pm 0.04$. The figures indicate that the resolving times t_2 and t_3 remained practically constant,

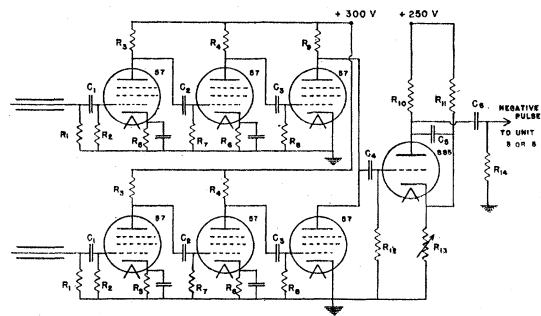


FIG 3. High resolving power double coincidence circuits (unit 4 or 6). $R_1 = 10^8$; $R_2 = R_3 = R_4 = R_7 = R_8 = 50,000$; $R_5 = R_6 = 10,000$; $R_9 = 150,000$; $R_{10} = 5,000$; $R_{11} = 25,000$; $R_{12} = 300,000$; $R_{13} = 10,000$ adjustable; $R_{14} = 50,000$; $C_1 = 0.00001$; $C_2 = 0.00001$ in unit 4, $= 0.00005$ in unit 6; $C_3 = 0.00005$; $C_4 = C_6 = 0.0001$; $C_5 = 0.1$. Resistance in ohms, capacity in μf .

TABLE I. Resolving time of three double coincidence circuits.

NUMBER OF HOURS	VOLTAGE ON COUNTERS	$2N_1N_2$ SEC. ⁻²	RANDOM COINCIDENCES IN SEC. ⁻¹ × 10 ³	t MICRO-SECONDS
<i>Unit 1</i>				
14.4	1065	4900	77 ± 2	15.7 ± 0.3
23.1	1065	4730	72 ± 1	15.2 ± 0.2
23.2	1035	4625	68 ± 1	14.7 ± 0.2
23.3	1035	1350	20.3 ± 0.5	15.0 ± 0.3
<i>Unit 4</i>				
23.1	1065	4730	8.6 ± 0.3	1.81 ± 0.07
23.2	1035	4625	8.9 ± 0.3	1.93 ± 0.07
23.3	1035	1350	2.5 ± 0.15	1.88 ± 0.12
<i>Unit 6</i>				
14.4	1065	4900	4.9 ± 0.3	1.00 ± 0.06
23.1	1065	4730	4.2 ± 0.2	0.90 ± 0.05
23.2	1035	4625	4.6 ± 0.2	0.99 ± 0.05
23.3	1035	1350	1.3 ± 0.15	0.97 ± 0.12

whereas the value of t_4 evidently decreased throughout the series of experiments. This variation is not surprising, as the characteristics of the tubes may have changed to some extent through more than 3000 hours of uninterrupted service. We shall use the following values of the resolving times (in microseconds):

$$t_2 = 15$$

$$t_3 = 1.95$$

$$t_4 = 0.95 \text{ for the Fe measurements}$$

$$t_4 = 0.76 \text{ for the Al measurements.}$$

Efficiency of the high resolving power circuits

The results of the present experiment are significant only if it has been ascertained that all truly coincident discharges (e.g., those produced by a single cosmic-ray particle) are registered by all three recorders 2, 3, and 4.

A simple procedure to check this point is the following. In unit 1, the Rossi tubes connected to counters *A*, *B* and *C* are disconnected, so that recorder 2 now counts processes of the type $(DE-F)$ at the rate of 900 per hour. All the rest of the circuit is left as in the actual experiment. Thus, recorders 3 and 4 each register the process $\{(DE-F)(DE)\}$.

If random coincidences (DE) are neglected, all recorders should register the same number of counts. Actually, recorders 3 and 4 were observed to lag behind recorder 2 at the rate of about 5 per hour (0.5 percent of the counts, against 15 to 25 percent in the actual experiments), which is the theoretical rate of random coincidences

(DE) recorded by circuit 1. Hence, we may safely assume that the circuits 4 and 6 do not miss any systematic coincident discharges. An experiment performed with a resolving time of 0.5 microsecond still showed full efficiency in coincidence recording.

The check described above was repeated every day, at least for a few minutes. During the whole series of the final experiments reported in the next section (over a period of three months) the apparatus functioned uninterruptedly, without requiring the slightest readjustment or replacement. All circuits were fed through a Raytheon voltage stabilizer.

3. RESULTS AND DISCUSSION

Measurement of the mean life

In order to measure the distribution in time of the particles emitted from the absorber, the number n_2 of $(ABCDE-F)$ processes was counted on recorder 2, and simultaneously the numbers n_3, n_4 of processes $\{(ABCDE-F)(DE)\}$, respectively, on recorder 3 and 4. Iron and aluminum were used as absorbers. Blank runs without absorber were interpolated.

The results are summarized in Table II.

The mean life τ of the disintegration process is calculated from the differences of the numbers of counts, according to the formula:

$$\exp\left(-\frac{t_3 - t_4}{\tau}\right) = \frac{n_2 - n_3}{n_2 - n_4}$$

and hence the result is not affected by instantaneous processes (such as the scattering of mesotrons) which contribute the same number of counts on the three recorders.

The small effect observed without absorber was not subtracted, since it is considerably larger than the expected random coincidence rate, and hence probably results from mesotrons disintegrating in the counter walls or in the support of the absorber, which, together, represent a thickness of about 4 g/cm² between counters *D* and *F*.

It is interesting to note that the presence of the Fe or Al absorber increased the counting rate n_2 by, respectively, 0.45 ± 0.05 and 0.30 ± 0.05 per hour. Now, from the difference

$n_2 - n_4$ and from $\tau = 1.5$ microseconds, we find for the total number of mesotron disintegrations which should be recorded between $t=0$ and $t = \infty$ the respective rate:

Fe	0.35 ± 0.07
Al	0.20 ± 0.06 .

Since these rates are not considerably lower than the ones observed, we may conclude that, even with Fe, most of the additional processes ($ABCDE - F$) produced by the absorber are due to disintegrations and not to scattered mesotrons. Experiments with a Pb absorber seem to indicate that this is not the case for such a heavy element.

In conclusion, the mean life of the disintegration process is about 1.5 microseconds, no significant difference being observed between the experiments with Al and Fe.

Number of disintegration electrons

To measure the absolute number of disintegration electrons, one must know: (a), the number of absorbed mesotrons; (b), the number of recorded disintegration electrons; (c), the probability that a disintegration electron will be recorded by the counters.

The absorption of disintegration electrons in iron would be exceedingly difficult to evaluate theoretically under the geometrical conditions prevailing in the experiments, and hence the measurements with an iron absorber are practically useless for the present purpose.

In aluminum, however, the range of the electrons should¹⁰ be about 6 cm if their energy is 40 Mev. Therefore, absorption should have little effect on the number of electrons that reach the counters, and the probability of detection can be roughly evaluated from purely geometrical factors. In the present case it also seems reasonable to assume that most of the anticoincidences

TABLE II. *Distribution in time of mesotron disintegrations.*

ABSORBER	HR.	n_2	n_3	n_4	$n_2 - n_4$	$n_2 - n_3$	τ MICROSEC.
None	311	106	102	102	4 ± 2	4 ± 2	—
10 cm Fe	602	473	417	360	113 ± 11	56 ± 8	1.4 ± 0.3
10 cm Al	514	331	301	268	63 ± 8	30 ± 6	1.6 ± 0.4

¹⁰ H. Bethe and W. Heitler, Proc. Roy. Soc. **A146**, 83 (1934).

produced by the absorber are due to stopped rather than to scattered mesotrons.

The number of mesotrons absorbed by the aluminum was directly determined by counting the anticoincidences ($ABCD - F$) with and without absorber. The results are summarized in Table III.

The geometrical probability for an electron to hit one of the counters E was evaluated graphically as 0.56. The effect of absorption

TABLE III. *Absorption of mesotrons in aluminum.*

ABSORBER	HOURS	ANTI-COINCIDENCES	ANTI-COINCIDENCES PER HOUR	DIFFERENCE
None	175	258 ± 16	1.48 ± 0.09	0.97 ± 0.15
10 cm Al	175	428 ± 21	2.45 ± 0.12	

should not be large, considering that the thickness of the absorber (2.5 cm) is much less than the average range of the electrons. We may, therefore, estimate the probability of detection of a disintegration electron as, roughly, 0.5.

By using this factor, we conclude that the counters E should record 0.48 ± 0.08 electron per hour, on the assumption that there is an electron for each absorbed mesotron. The actual rate recorded (extrapolated at the time zero) is only 0.20 ± 0.06 per hour. This gives as the number of disintegration electrons per mesotron 0.42 ± 0.15 .

Interpretation of results

The value obtained for the mean life of slow mesotrons in dense materials agrees, as satisfactorily as can be expected from the accuracy of the measurements, with the proper lifetime of fast mesotrons in the atmosphere. Actually, our value is lower than the one found by Rossi and Hall,¹ but higher than that reported by the Duke University group.²

The proper lifetime for fast mesotrons is deduced from the measurement of the disintegration probability per unit length of path

$$w = \mu / p\tau,$$

p being the momentum and μ the mass of the mesotron. It is gratifying to find that two so widely different methods have yielded consistent

results. This agreement, and the substantially equal values for the mean life measured in Al and Fe, render it plausible to assume that these values represent the mean life for spontaneous decay.

The present experiments seem to indicate a number of disintegration electrons per mesotron definitely smaller than unity. The statistical error is certainly large, and systematic errors are not to be excluded, although it seems rather unlikely that they might be large enough to account for a discrepancy by more than a factor two.

The results, however, are in agreement with the assumption that only half of the mesotrons undergo free decay. Since the analysis of mesotron tracks in a magnetic field has shown that there are about as many positive as negative

mesotrons, or a small excess of positive,¹¹ the result found is what should be expected if only mesotrons of one sign (positive) undergo free decay. Actually, if, according to the calculations of Tomonaga and Araki, reactions with nuclear particles are much more probable than spontaneous disintegration for negative mesotrons, then we should only record an electron for each positive mesotron absorbed. The nuclear reactions produced by negative mesotrons will probably lead to excited states of nuclei and eventually give rise to electrons through processes of β -decay. It is exceedingly unlikely, however, that such particles could be emitted with sufficient energy and within a sufficiently short time to be registered in the present experiments.

¹¹ See H. Jones, *Rev. Mod. Phys.* **11**, 235 (1939), also for earlier literature.

Theory of Nuclear Surface Energy

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The eigenvalues of a free particle in a spherical potential well of finite depth are computed and used to calculate the surface energy of nuclear systems. No assumptions are made concerning the specifically nuclear forces. For depth and radius $56mc^2$ and $\frac{1}{2}A^{1/2}/mc^2$, respectively, the computed surface energy is about two-thirds the empirical value. A well of infinite depth yields a surface energy more than double the empirical value. One-dimensional and cubical wells are discussed for the purpose of orientation.

INTRODUCTION

THE empirical packing fraction curve¹ and the phenomenon of fission² require the existence of a nuclear surface energy having the magnitude $26A^{2/3}mc^2$ within limits of perhaps ± 10 percent. There exists no adequate theoretical calculation of this quantity, although estimates have been obtained by Weizsäcker³ and Bethe.⁴ A complete theoretical discussion is not possible without the use of special assumptions

about the nuclear forces, but it is clear without calculation that the specifically nuclear forces must make a positive contribution to the surface energy. If γ is written for the coefficient of $A^{2/3}$ in the semi-empirical formula for nuclear energies, and γ_K , γ_P for the contributions from the kinetic and potential energy operators, respectively, these remarks may be summarized in the relations

$$\gamma = \gamma_K + \gamma_P \sim 26mc^2, \quad \gamma_K < 26mc^2. \quad (1)$$

The difficulties and uncertainties barring a theoretical determination of γ_P are not present to the same degree for γ_K . It is, therefore, desirable to obtain a first approximation for γ_K

¹ A. J. Dempster, *Phys. Rev.* **53**, 869 (1938).

² N. Bohr and J. A. Wheeler, *Phys. Rev.* **56**, 426 (1939).

³ C. F. v. Weizsäcker, *Zeits. f. Physik* **96**, 431 (1935).

⁴ H. A. Bethe and R. F. Bacher, *Rev. Mod. Phys.* **8**, 82 (1936).