Advanced Particle Physics 1

Electromagnetic Interactions

L6 – Introduction to Gauge Theories

(http://dpnc.unige.ch/~bravar/PPA1/L6)

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Gauge Theories

Is there a symmetry principle powerful enough to dictate the form of the interaction?

The form of the interaction in QED is known from classical theory of Maxwell et *al*.
There are no classical counterparts for the Strong and Weak Interactions.
In general, guess a suitable form of the interaction and confront it with experiment (particle spectrum, known symmetries and conservation laws, cross-sections, decays, ...)

Quite generally, the form of interaction is restricted by requiring
- Lorentz invariance
- locality (the fields are evaluated at same space-time point)
- renormalizability

By demanding the local phase invariance of the theory under some internal symmetry transformation we are led to introduce gauge invariant fields (i.e. the gauge bosons) that mediate the interaction.
The resulting equations seem inevitable.

QED is a renormalizable gauge field theory and
renormalizable theories are gauge field theories, i.e. possessing local phase invariance.
Elementary particle physics is almost exclusively concerned with such theories:
QCD and GWS are both gauge filed theories, remarkable generalizations of QED.

Today we will discuss how QED is based on a small number of abstract but plausible physics principles:
the invariances of a system of particles.

We have already established the equations of motion allowed by covariance under Lorentz transformations. This not only has provided a valid description of relativistic particles, but also the concepts of antiparticles and spin.

All the discussions in previous chapters are based on the wave function of a single particle and its interaction vertex mediated by a single photon.

The classical analogy is the kinematics of a point mass, and its dynamics.
The symmetries of particles and their interactions concern, on the other hand, systems of many particles.

In classical physics this is discussed by the Lagrangian formalism.

First we have to review the Lagrangian formalism for classical continuous fields, and the role played by symmetry (Noether's theorem).
Then we will examine the role of phases in quantum physics.
Finally we will derive QED by requiring gauge invariance or local phase invariance of the theory.

Lagrangians

Classical mechanics for point particles or continuous systems can be expressed in terms of a Lagrangian $L = L(q(t), \dot{q}(t), t)$, which depends on the generalized coordinates $q_i(t)$, the generalized velocities $\dot{q}_i(t)$, and possibly explicitly on time *t*:

$$\boxed{L = T - V}$$

i.e. the kinetic energy minus the potential energy.
The force enters at the derivative of the potential.
The path integral of the Lagrangian gives the classical action

$$\boxed{S = \int_{t_1}^{t_2} \mathrm{d}t \; L(q, \dot{q})}$$

and the equations of motion follow from the Hamilton principle of minimal action $\delta S = 0$:

$$\boxed{\delta S = \delta \int_{t_1}^{t_2} \mathrm{d}t \; L(q, \dot{q}) = 0}$$

with boundary conditions $\delta q_i(t_1) = \delta q_i(t_2) = 0$.
If *L* does not depend explicitly on time, we obtain the equation of motion from the Euler-Lagrange equations

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0}$$

For instance $L = \frac{1}{2} m \ddot{x} - V(x) \quad \rightarrow \quad m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x} = F$ (Newton's law)

Continuous Systems

The extension of the Lagrangian formalism of discrete coordinates $q_i(t)$ to continuous fields $\phi(\mathbf{x}, t)$ is strigtforward.
We can base a field theory on a Lagrangian density $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x), x_\mu)$$

which in general depends on the field $\phi(x)$, its four-gradient $\partial_\mu \phi(x)$, and space-time x_μ.
A field is just a property of the system that can exist in all space points simultaneously, for example the electromagnetic field, or the probability amplitude (wave function) of a particle in Quantum Mechanics.

In the Lagrangian formalism a field can be viewed as a generalized coordinate $q_i = \phi(x_i)$ at each point of the space-time x_i.
This gives a description of a closed system with an infinite number of degrees of freedom.

The action integral becomes

$$S = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\vec{x}\ \mathcal{L}(\phi, \partial_\mu \phi)$$

The Hamilton principle $\delta S = 0$ with boundary conditions $\delta\phi(x_1) = \delta\phi(x_2)$ still applies and can be used to obtain the equations of motion for the field $\phi(x)$.

In the following we will simply call the Lagrangian density the Lagrangian.

Consider a variation of the field $\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta\phi(x)$.

Then

$$\delta S = \int \mathrm{d}^4x \; \delta\mathcal{L}(\phi, \partial_\mu \phi) = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right]$$

Noting that $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$ and integrating the second term by parts (we assume that $\mathcal{L}$ does not depend explicitly on x_μ)

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi$$

The Hamilton principle (the action is stationary, $\delta S = 0$) gives the Euler-Lagrange equations for ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

which yields the equations of motion for the field ϕ.

If the Lagrangian density $\mathcal{L}$ is a scalar under the Lorentz transformations, the equations of motion for the field ϕ are Lorentz covariants.

The field itself can be a scalar, a spinor, a vector, ...

For example for a free scalar field ϕ(x), the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2$$

gives the Klein-Gordon equation $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$.

The Electro-Magnetic Field

Classical electrodynamics can be formulated as a Lagrangian theory.
The electromagnetic fields **E** and **B** can be expressed in terms of a potential $A^\mu = (A^0, \mathbf{A})$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\partial \mathbf{A} / \partial t - \nabla A^0$$

Introduce the (antisymmetric) filed tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

with components

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = \varepsilon^{ijk} B^k$$

Each component of the vector potential A^μ satisfies a Klein-Gordon equation for massless particles

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu = \Box A^\nu = 0$$

and can be identified with the photon field.

The conventional Lagrangian density $\mathcal{L}$ for electromagnetism is

$$\mathcal{L} = \frac{1}{2}(E^2 - B^2) - \rho V + \vec{J} \cdot \vec{A}$$

or in terms of $F^{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J_\mu \cdot A^\mu$$

The first term is the kinetic term and describes the motion of a free photon for $J_\mu = 0$, while the second term correspond to the interaction between the photon and the electron current.

The Maxwell equations follow from the Euler-Lagrange equations ($\delta S = 0$):

homogeneous $$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$$

inhomogeneous $$\partial_\mu F^{\mu\nu} = -J^\nu \quad \Rightarrow \quad \partial_\mu J^\mu = 0$$

We see therefore that the Lagrangian formalism can be used to describe not only a system of free particles and fields, but can also their interactions.

Proof

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial V} = -\rho, \quad \frac{\partial \mathcal{L}}{\partial(\partial_x V)} = -E_x, \quad \ldots, \quad \frac{\partial \mathcal{L}}{\partial(\partial_t V)} = 0$$

$$\Rightarrow \quad \partial_k \frac{\partial \mathcal{L}}{\partial(\partial_k V)} = -\nabla \cdot E = -\rho$$

$$\frac{\partial \mathcal{L}}{\partial A_x} = J_x, \quad \frac{\partial \mathcal{L}}{\partial(\partial_x A_x)} = 0, \quad \frac{\partial \mathcal{L}}{\partial(\partial_t A_x)} = -E_x, \quad \frac{\partial \mathcal{L}}{\partial(\partial_y A_x)} = B_x$$

$$\Rightarrow \quad \partial_k \frac{\partial \mathcal{L}}{\partial(\partial_i A^j)} = \nabla \times B = J$$

Gauge Invariance in Classical EM

Note that we have taken the potentials as the basic fields of the theory, not **E** and **B**.
The potentials, however, are not unique, since a gauge transformation of the form

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x) = (\phi + \partial\Lambda/\partial t, \vec{A} - \nabla\Lambda)$$

leaves the Maxwell equations invariant ($\Lambda(x)$ is an arbitrary differentiable scalar field).
Because of the gauge ambiguity, the potential A^μ, corresponding to particular **E** and **B** fields, is not uniquely defined,
i.e. the potential contains "too much" information and it is not observable!

The electromagnetic current $\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = -J^\nu$

however is conserved $\partial_\nu J^\nu = -\partial_\nu \partial_\mu F^{\mu\nu} = 0$

Under a gauge transformation the action S acquires an additional term

$$\Delta S = -\int d^4x \; J_\mu \partial^\mu \Lambda = \int d^4x \; (\partial^\mu J_\mu)\Lambda$$

ΔS is zero for arbitrary Λ if, and only if

$$\partial_\mu J^\mu = \partial^\mu J_\mu = 0$$

Thus the gauge invariance of the action requires, and follows from, the conservation of the electric charge.

Symmetry Transformation

The connection between symmetries and laws of conservation is well known in classical and quantum mechanics $\rightarrow$ Noether's theorem.

Noether's theorem tells us when conserved quantities exists:
if a conserved quantity is observed, there must be an associated symmetry;
and if there is a symmetry there must be a conserved quantity.
This tells us how to build the Lagrangian, i.e. how to relate observed symmetries and conservation laws in the structure of the theory.

Giving the Lagrangian is equivalent to define the theory.

Invariance under a continuous transformation implies an associated conservation law (whenever there is an invariance there is a corresponding conserved quantity):

translational invariance ⇔ linear momentum conservation

time translation invariance ⇔ energy conservation

rotational invariance ⇔ angular momentum conservation

Internal Symmetry Transformation

Beyond the transformations of external parameters and coordinates of space and time, let's extend the discussion to internal symmetries, which concern the fields themselves: instead of considering the transformations $\phi(x) \rightarrow \phi(x')$ we will consider $\phi(x) \rightarrow \phi'(x)$.

Consider a system of two real scalar fields ϕ_1 and ϕ_2, having the same mass m

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \phi_1 \partial^\mu \phi^1 - m^2 \phi_1^2\right] + \frac{1}{2}\left[\partial_\mu \phi_2 \partial^\mu \phi^2 - m^2 \phi_2^2\right]$$

We can combine ϕ_1 and ϕ_2 in a single complex field ϕ (ϕ and ϕ^* are independent fields):

$$\phi = \left(\phi_1 + i\phi_2\right) / \sqrt{2}$$

$$\phi^* = \left(\phi_1 - i\phi_2\right) / \sqrt{2}$$

Then $\mathcal{L}$ becomes

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

Nothing fixes the particular direction of ϕ_1 and ϕ_2 .
We could have equally started with two fields ϕ_1 and ϕ_2 that were "rotated" by an angle α

$$\phi_1' = \phi_1 \cos\alpha + \phi_2 \sin\alpha$$

$$\phi_2' = -\phi \sin\alpha + \phi_2 \cos\alpha$$

or in exponential form $\phi' = e^{-i\alpha}\phi$ and $\phi^{*\prime} = e^{+i\alpha}\phi^*$

There is clearly no change in $\mathcal{L}$ since it depends on $\phi^*\phi$ ($\delta\mathcal{L}=0$).

Assume that α is infinitesimal $\phi' \approx (1-i\alpha)\phi = \phi - i\alpha\phi \equiv \phi + \delta\phi$

The change in ϕ and ϕ^* is then $\delta\phi = -i\alpha\phi$ and $\delta\phi^* = +i\alpha\phi^*$

Let's calculate explicitly the change in the Lagrangian (we know that it is 0 …)

$$\delta\mathcal{L} = \delta\phi\frac{\partial\mathcal{L}}{\partial\phi} + \delta\left(\partial^\mu\phi\right)\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)} + \left(\phi\to\phi^*\right)$$

The second term can be rewritten as

$$\delta\left(\partial^\mu\phi\right)\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)} = \partial^\mu\left\{\delta\phi\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\right\} - \delta\phi\left\{\partial^\mu\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\right\}$$

and

$$\delta\mathcal{L} = \delta\phi\left\{\frac{\partial\mathcal{L}}{\partial\phi} - \partial^\mu\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\right\} + \left(\phi\to\phi^*\right) + \partial^\mu\left\{\delta\phi\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)} + \delta\phi^*\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi^*\right)}\right\}$$

The first term vanishes (don't need to use Euler-Lagrange, just Klein-Gordon) and

$$\delta\mathcal{L} = \partial^\mu\left\{\delta\phi\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi\right)} + \delta\phi^*\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\phi^*\right)}\right\}$$

This is a general result, which does not dependent on the details of our transformation. The variation in $\mathcal{L}$ can be written as the divergence of the quantity in the brackets.

Let's calculate explicitly the quantity in the brackets:

$$\delta\phi \frac{\partial\mathcal{L}}{\partial\left(\partial^{\mu}\phi\right)} + \delta\phi^{*} \frac{\partial\mathcal{L}}{\partial\left(\partial^{\mu}\phi^{*}\right)} = -i\alpha\,\phi\,\partial_{\mu}\phi^{*} + i\alpha\,\phi^{*}\partial_{\mu}\phi = i\alpha\left(\partial_{\mu}\phi\,\phi^{*} - \partial_{\mu}\phi^{*}\phi\right)$$

and define $\boxed{J_{\mu} = i\left(\partial_{\mu}\phi\,\phi^{*} - \partial_{\mu}\phi^{*}\phi\right)}$ the corresponding current.

Then the variation of $\mathcal{L}$ can be written as $\delta\mathcal{L} = \alpha\,\partial^{\mu}J_{\mu}$

and $\boxed{\delta\mathcal{L} = 0 \Leftrightarrow \partial^{\mu}J_{\mu} = 0}$

The quantity in brackets behaves like a conserved current, i.e. its four-divergence is 0.
Whenever the Lagrangian is invariant under a set of continuous transformations,
a divergenceless current arises.
This leads to an explicitly conserved charge *Q*.
Integrate over d^3x

$$\partial_{\mu}J^{\mu} = 0 \qquad \Leftrightarrow \qquad \int \partial_{\mu}J^{\mu}\mathrm{d}^3x = \int \partial_{0}J^{0}\mathrm{d}^3x + \int \partial_{i}J^{i}\mathrm{d}^3x = 0$$

The second term vanishes (surface integral at infinity) and

$$\boxed{\frac{\partial}{\partial t}\int J^{0}\mathrm{d}^3x = 0}$$

The associated charge $Q = \int J^{0}\mathrm{d}^3x$ is conserved in the sense that $dQ/dt = 0$.

Whenever there is a conserved current, there is also a conserved charge and vice versa

Lagrangians in Particle Physics

Formulate particle physics by giving the Lagrangian density $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$.

The equations of motions follow from variational principles (Euler-Lagrange equation).
Example: spin-1/2 fermion of mass m (Dirac Lagrangian)

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \quad \rightarrow \quad \left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$$

Using QFT rules all observables can be calculated, i.e. the Lagrangian defines the theory.
The kinetic energy part describes the motion of free particles → $\mathcal{L}_{free}$.
The potential energy part specifies the theory, i.e. the fundamental interactions of the theory (the forces) → $\mathcal{L}_{int}$.

Why Lagrangians?

The Hamiltonian corresponds to a conserved quantity (the total energy) while the Lagrangian does not. Hamiltonians however are not Lorentz invariant.
The Lagrangian is a single real function that determines the dynamics, and must be a scalar invariant under Lorentz transformations, since the action is invariant.
Lorentz invariance → all predictions of the theory are Lorentz invariant.

Symmetry transformation of the fields readily expressed via the invariance of *L*.
If the Lagrangian is invariant under some transformation (more precisely the action), then there is a corresponding conserved current (Noether's theorem).

Lagrangians and Feynman Rules

Let's formulate explicitly the connection between the Lagrangian formalism and our perturbative calculation method based on the Feynman rules: each Lagrangian density $\mathcal{L}$ corresponds to a set of Feynman rules.
The terms in $\mathcal{L}$ correspond to propagators (particles) and to vertex factors (interactions between particles). To find them we use the following recipe :

i) The propagators follow from the terms that are quadratic in the fields and their derivatives, for example:

$$\frac{1}{2}\left(\partial_\mu \phi\right)^2; \quad \frac{1}{2} m^2 \phi^2; \quad i\bar{\psi}\gamma_\mu \partial^\mu \psi; \quad F_{\mu\nu}F^{\mu\nu}$$

We can obtain the propagators explicitly via the equations of motion by the Euler-Lagrange equations.
We recognize here also a mass term 1/2 m^2 ϕ^2 .

ii) All other terms in the Lagrangian density, for example

$$\mathcal{L}_{int} = -J_\mu A^\mu = iq\bar{\psi}\gamma_\mu \psi A^\mu$$

correspond to the interaction terms.
The coefficients of the terms in $\mathcal{L}_{int}$ that contain the interacting fields, i.e. $-iq\gamma_\mu$, are the vertex factors.

We will not prove these conjectures formally (→ quantization of the fields).

Global Phase Transformations

Let's start with the example of the Lagrangian density of a free fermion:

$$\mathcal{L} = \bar{\psi}\left(i\gamma_\mu\partial^\mu - m\right)\psi$$

Using the Euler-Lagrange equations we obtain the equations of motion for the field ψ, i.e. the Dirac equation:

$$\left(i\gamma_\mu\partial^\mu - m\right)\psi = 0$$

This Lagrangian is invariant under global phase transformations of the field ψ:

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x)$$

with α a constant phase.

The global characterization concerns the parameter α, which should not depend on x_μ. By substitution we obtain

$$\mathcal{L}' = e^{-i\alpha}\bar{\psi}\left(i\gamma_\mu\partial^\mu - m\right)e^{i\alpha}\psi$$

Since $\partial^\mu e^{i\alpha}\psi = e^{i\alpha}\partial^\mu\psi$ the Lagrangian density is indeed invariant for a global phase α

$$\mathcal{L}' = \bar{\psi}'\left(i\gamma_\mu\partial^\mu - m\right)\psi' = \bar{\psi}\left(i\gamma_\mu\partial^\mu - m\right)\psi = \mathcal{L}$$

The global phase transformations $U(1) = \mathbf{1}e^{i\alpha}$ with a single real parameter form an abelian unitary Lee group.

According to Noether's theorem, the symmetry ("invariance") of the Lagrangian density under the global group U(1) should lead to a conservation law for some observable.
To find it, we consider an infinitesimal transformation U(1):

$$\psi \rightarrow (1+i\alpha)\psi \quad \text{and} \quad \bar{\psi} \rightarrow (1-i\alpha)\bar{\psi}$$

so $\delta\psi = i\alpha\psi$ and $\delta(\partial_\mu\psi) = i\alpha\partial_\mu\psi$.

The invariance of the Lagrangian density requires

$$0 = \partial\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\delta\left(\partial_\mu\psi\right) + \left(\psi \rightarrow \bar{\psi}\right)$$

$$= \frac{\partial\mathcal{L}}{\partial\psi}\left(i\alpha\psi\right) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\left(i\alpha\partial_\mu\psi\right) + \left(\psi \rightarrow \bar{\psi}\right)$$

$$= i\alpha\left[\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\right]\psi + i\alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\psi\right) + \left(\psi \rightarrow \bar{\psi}\right)$$

The first term is null because the Euler-Lagrange equations and the condition for invariance is reduced to

$$\partial\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\psi - \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar{\psi}\right)}\bar{\psi}\right)$$

The term in the parentheses gives $2i\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right)$.

This corresponds to the conservation law for the electromagnetic current density,

$$j^{\mu} = -e\bar{\psi}\gamma^{\mu}\psi$$

in other words the continuity equation, $\partial_{\mu} j^{\mu} = 0$.

This is very general: whenever a physical system is invariant under some transformations it leads to conserved quantities.
For a system described by a Lagrangian, any continuous symmetry which leaves invariant the action, leads to the existence of a conserved current.

It is always possible to define a charge Q

$$Q = \int j^{0} \mathrm{d}^{3}x$$

which is conserved in the sense that $\mathrm{d}Q / \mathrm{d}t = 0$.

This significant conclusion comes from the "simple" requirement that the Lagrangian density for fermions (and of course also for scalars) is invariant under the global phase transformations of the U(1) group.
We can say that global gauge invariance is the **theoretical origin** of the charge conservation.

For historical reasons these transformations are also called
global **gauge** transformations of the U(1) group.
The invariance means also that the absolute phases are not observable.

Phase Invariance in Quantum Mechanics

Suppose that we know the Schrödinger equation but not the laws of electrodynamics.
Can we guess Maxwell's equations from a gauge symmetry principle?
Yes! But …

QM observables are unchanged under global phase transformations of the wave function

$$\psi(x) \to \psi'(x) = e^{i\vartheta}\psi(x)$$

The absolute phase of the wave function cannot be measured and relative phases (like in interference experiments) are unaffected by this transformation.
Can we chose freely the phase in Geneva and Paris?
In other words, is QM invariant under local phase transformations?

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$$

Yes! But …

QM equations always involve derivatives

$$\partial_\mu \psi(x) \to \partial_\mu \psi'(x) = e^{i\alpha(x)}\left[\partial_\mu \psi(x) + i\left(\partial_\mu \alpha(x)\right)\psi(x)\right]$$

The additional term spoils the local phase invariance. Note that $\partial_\mu \alpha(x)$ is a vector!

Local phase invariance can be restored if the equations of motion and observables involving derivatives are modified by introducing a vector field A^μ (the EM field).

The gradient ∂_μ is replaced everywhere by the covariant derivative

$$\partial_\mu \rightarrow D_\mu = [\partial_\mu + iqA_\mu(x)]$$

such that also the covariant derivative D_μ transforms in the same was as Ψ

$$D_\mu \psi(x) \rightarrow D'_\mu \psi'(x) = e^{i\alpha(x)} D_\mu \psi(x)$$

Then quantities such as $\psi^*(x) D_\mu \psi(x)$ are invariant under local phase transformations.

Let's find out how the field A^μ transforms by writing out explicitly the various terms

$$D^{\mu\prime} \psi'(x) = (\partial^\mu + iqA^{\mu\prime}) e^{i\alpha(x)} \psi = e^{i\alpha(x)} (\partial^\mu + iqA^\mu)\psi$$

and solve for A^μ

$$iqA^{\mu\prime} e^{i\alpha(x)} \psi = -\partial^\mu \left(e^{i\alpha(x)} \psi\right) + e^{i\alpha(x)} \partial^\mu \psi + iqe^{i\alpha(x)} A^\mu \psi = -\partial^\mu \left(e^{i\alpha(x)}\right)\psi + iqe^{i\alpha(x)} A^\mu \psi$$

Since each term acts on an arbitrary state Ψ, we can drop Ψ and

$$A^\mu(x) \rightarrow A^{\mu\prime}(x) = A^\mu(x) - \frac{1}{q}\partial^\mu \alpha(x)$$

We reestablished the invariance under local phase transformations at the price of introducing a vector field A^μ which gives a local interaction term $\Psi^* qA^\mu \Psi$, that will be constructed to be electromagnetism.

The required transformation law for A^{μ} is precisely the same as in classical EM, i.e. up to a gradient of a scalar field $\partial_{\mu}\alpha(x)$, and the covariant derivative corresponds to the minimal substitution $\mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}$ of EM.

The form of the coupling between the EM field and matter is suggested by $D_{\mu}\Psi \rightarrow \Psi^{*} qA^{\mu}\Psi$.

We used a local gauge invariance as dynamical principle which led us to modify the equations of motion, i.e. we have built the interaction term D_{μ} and arrived at an interacting theory.

Note that Maxwell by imposing local charge conservation was led to modify Ampere’s law by the addition of the displacement current dE/dt.

Aharonov-Bohm Effect

Electrodynamics is invariant under gauge transformations of the vector potential

$$A^{\mu}(x) \rightarrow A^{\mu}(x) + \partial^{\mu}\Lambda(x) = (\phi + \partial\Lambda / \partial t, \vec{A} - \nabla\Lambda)$$

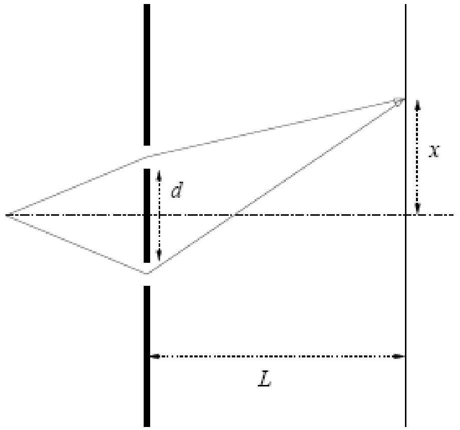
without affecting any physical laws,
which implies that the potential $A^{\mu}(x)$ is not a physical observable
(**E**, **B**, $F^{\mu\nu}$ are gauge invariant, A^{μ} is not, only potential differences are observable).

Are potentials physical or just calculational tools?
The vector potential does have a significance in quantum physics, as shown by Aharonov and Bohm (1959).

Let's imagine a two split experiment (i.e. split a coherent beam of charged particles in two parts), and let's observe the interference pattern on a far screen.

The wavefunction at a given point on the screen has the form

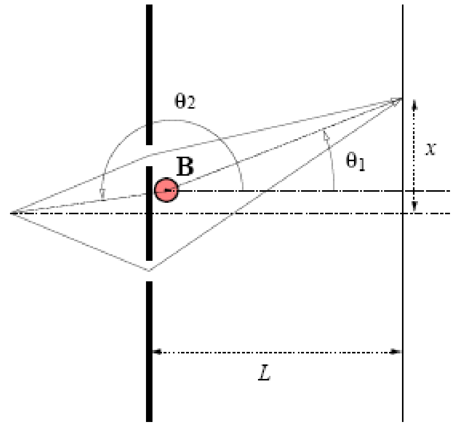
$$\psi \sim \psi_0 \left[1 + \exp\left(i\delta\varphi\right)\right]$$ with $$\delta\varphi = \frac{2\pi}{\lambda}(d_2 - d_1) = \frac{2\pi}{\lambda}\frac{xd}{L}$$



Young's experiment

Now introduce an infinite solenoid behind the slits.
There is no magnetic field outside of the solenoid (B = 0),
B is confined inside the solenoid, however **A** $\neq$ 0 everywhere

$$\vec{A} = \begin{cases} \frac{1}{2} B r \hat{\varphi} & r < R \\ \frac{1}{2} B \frac{R^2}{r} \hat{\varphi} = \frac{\Phi_B}{2\pi r} \hat{\varphi} & r > R \end{cases}$$



Aharonov-Bohm's experiment

What happens to a non-relativistic charged particle moving through a static vector potential that corresponds to a vanishing magnetic field?

If $\Psi_0(\mathbf{x},t)$ is the solution of the Schrödinger equation for A = 0,
the solution of the Schrödinger equation in the presence of the vector potential **A**

$$\frac{\left(-i\hbar\nabla - q\mathbf{A}\right)^2}{2m}\psi(\vec{x},t) = i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t}$$

is $\psi(\vec{x},t) = \psi_0(\vec{x},t)\exp(iS/\hbar)$ with $S = q\int d\vec{x}\cdot\mathbf{A}$

The phase shift experienced by the particle is the change in its classical action.
The fact that the new solution differs from the unperturbed one simply by a phase factor implies that there is no change in any physical result.

By analogy with the Young's experiment, the "perturbed" wave function is

$$\psi(\vec{x},t) = \psi_{0,1}(\vec{x},t)\exp(iS_1/\hbar) + \psi_{0,2}(\vec{x},t)\exp(iS_2/\hbar)$$

The phase difference at the screen between the two paths becomes

$$\exp\left(i\delta\varphi + iq\int_2 \vec{A}\cdot \mathrm{d}\vec{x} - iq\int_1 \vec{A}\cdot \mathrm{d}\vec{x}\right) = \exp\left(i\delta\varphi + iq\oint \vec{A}\cdot \mathrm{d}\vec{x}\right)$$

The interference of the two components of the recombined beam will depend on the phase difference

$$\frac{S_1 - S_2}{\hbar} = \frac{q}{\hbar}\oint \mathrm{d}\vec{x}\cdot \mathbf{A} = \frac{q}{\hbar}\Phi_B$$

because the two beams followed different paths through the potential **A**.

The result is gauge independent, since $\oint \vec{\nabla}\Lambda \cdot \mathrm{d}\vec{x} = 0$

Since it is not possible to eliminate **A** in the empty space outside of the solenoid with a gauge transformation, the phase shift $\Delta\varphi_B = q\Phi_B$ becomes observable.
The vector potential does induce a physical observable effect.
This implies that the link between the phase transformation of the electron wave function and the gauge degree of freedom of the electromagnetic field is fundamental and goes beyond the classical predictions.
The Aharonov-Bohm effect has been confirmed experimentally in 1986.

QED: Dirac + EM Fields

We start with the Lagrangian for a free Dirac field

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^{\mu}\partial_{\mu} - m)\psi(x)$$

The EM field is introduced as in classical physics via the minimal substitution p → p – qA :

$$\partial_{\mu} \rightarrow D_{\mu} = [\partial_{\mu} + iqA_{\mu}(x)]$$

where A_{μ} is the electromagnetic potential.
We assume that this substitution introduces correctly the EM field into the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = q\gamma^{\mu}A_{\mu}(x)\psi(x)$$

The resulting Lagrangian acquires an interaction term $\mathcal{L}_{int}$

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^{\mu}D_{\mu} - m)\psi(x) = \mathcal{L}_0 - q\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x) = \mathcal{L}_0 + \mathcal{L}_{int}$$

The interaction term $\mathcal{L}_{int}$ couples the conserved current

$$j^{\mu}(x) = q\bar{\psi}(x)\gamma^{\mu}\psi(x)$$

to the electromagnetic field A_{μ}. *q* is the coupling constant to be determined by the exp.'t.
To complete the Lagrangian we add a term $\mathcal{L}_{rad}$ describing the radiation field

$$\mathcal{L}_{rad} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

with $F_{\mu\nu}$ the EM energy-momentum tensor $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$

Only the EM fields **E** and **B** have physical significance, not the potential A_μ itself, therefore the theory must be invariant under gauge transformations of the potentials

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{q}\partial_\mu \alpha(x)$$

where $\alpha(x)$ is an arbitrary real scalar differentiable function.

Before quantum theory this step could be argued to be a mathematical reformulation of Maxwell classical EM theory with no physical consequences.

The resulting Lagrangian

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \bar{\psi}(x)\gamma^\mu \psi(x)\partial_\mu \alpha(x)$$

however, is not invariant.

Invariance can be restored by demanding that the Dirac fields transform as

$$\psi(x) \to \psi'(x) = \psi(x)e^{-iq\alpha(x)}$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)e^{+iq\alpha(x)}$$

i.e. undergo a local phase transformation.

We started by introducing the EM interactions in the simplest way p → p – qA and required that the resulting Lagrangian is invariant under gauge transformations of the EM potential A_μ. This requires the local phase invariance of the Dirac fields. Now that we have identified a powerful invariance principle, we can proceed the other way by requiring that the Lagrangian is invariant under local phase transformations.

Gauge theory: any theory invariant under such coupled transformations. QED is the simplest example of such theories.

Gauge Fields

Let's start by requiring the invariance of the free Lagrangian $\mathcal{L}_0$

$$L_0 = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

under global phase transformations

$$\psi(x) \to \psi'(x) = \psi(x)e^{-i\alpha}$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)e^{+i\alpha}$$

$\mathcal{L}_0$ is invariant and this invariance ensures that current and charge are conserved:

$$j^\mu(x) = q\bar{\psi}(x)\gamma^\mu\psi(x) \qquad Q = q\int \mathrm{d}^3x\, \psi^\dagger(x)\psi(x)$$

Next we demand invariance under more general local phase transformations

$$\psi(x) \to \psi'(x) = \psi(x)e^{-iq\alpha(x)}$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)e^{+iq\alpha(x)}$$

The resulting Lagrangian

$$\mathcal{L}_0 \to \mathcal{L}_0' = \mathcal{L}_0 - q\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\alpha(x)$$

is not invariant (not a surprise!).

To restore the invariance of $\mathcal{L}_0$ we add an interaction term $\mathcal{L}_{int}$ by associating matter fields to the gauge field A_μ, which must transforms according to (A_μ itself is not gauge invariant!)

The interaction between matter and gauge fields is introduced via the minimal substitution in the free Lagrangian $\mathcal{L}_0$ by replacing the ordinary derivative with the covariant derivative

$$D_\mu \psi(x) = [\partial_\mu + iqA_\mu(x)]\psi(x)$$

The free Lagrangian transforms into

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) = \mathcal{L}_0 - q\bar{\psi}(x)\gamma^\mu \psi(x) A_\mu(x) = \mathcal{L}_0 + \mathcal{L}_{int}$$

where $\mathcal{L}_{int}$ describes the interaction between the Dirac field and the gauge field A_μ, known also as minimal gauge interaction.

The covariant derivative transforms in the same way as the Dirac fields

$$D_\mu \psi(x) \rightarrow D'_\mu \psi'(x) = e^{-iq\alpha(x)} D_\mu \psi(x)$$

provided that the gauge field A_μ transforms according to (A_μ itself is not gauge invariant!)

$$\left(\partial^\mu - iqA^{\mu\prime}\right) e^{-i\alpha(x)} \psi = e^{-i\alpha(x)} \left(\partial^\mu - iqA^\mu\right)\psi$$

And solving for A_μ

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{q}\partial_\mu \alpha(x)$$

Hence the resulting Lagrangian is invariant.

To complete the Lagrangian we add a term $\mathcal{L}_{rad}$ to describe the free gauge field

$$\boxed{\mathcal{L}_{rad} = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x)}$$

(for completeness, one would need to show that also this term is gauge invariant)

Finally, the resulting QED Lagrangian is

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(x)(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) - q\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x) - \frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x)}$$

To summarize: by requiring local gauge invariance (local phase invariance)
of the Dirac fields,
we are led to introduce a gauge field A_{μ} to preserve the invariance of the resulting Lagangian. By doing so we developed the full QED Lagrangian.

Can try to extend the gauge symmetry principle (local phase invariance) to other forces …

Generalization

Suppose we want to build a theory, which is invariant under some transformation *U*(x) (the transformation group U in general is non-abelian)

$$\psi'(x) = U(x)\psi(x)$$

We define the covariant derivative

$$D^\mu = \partial^\mu - igA^\mu(x)$$

and introduce the interacting vector field $A^\mu(x)$ to make the theory invariant.
g is the coupling constant to be determined from the experiment.
We want that the covariant derivative transforms in the same way as the spinor fields

$$D^{\mu\prime}\psi' = U(x)\left(D^\mu\psi\right)$$

Explicitly

$$\left(\partial^\mu - igA^{\mu\prime}\right)U\psi = U\left(\partial^\mu - igA^\mu\right)\psi$$

and solve for A^μ to obtain the transformation properties of the vector field $A^\mu(x)$

$$-igA^{\mu\prime}U\psi = -\partial^\mu\left(U\psi\right) + U\partial^\mu\psi - igA^\mu\psi = -\left(\partial^\mu U\right)\psi - igUA^\mu\psi$$

Since each term acts on an arbitrary state ψ (and *U* is not necessarily abelian)

$$A^{\mu\prime} = -\frac{i}{g}\left(\partial^\mu U\right)U^{-1} + UA^\mu U^{-1}$$

Gauge Theories

Is there a symmetry principle powerful enough to dictate the form or the interaction?

The form of the interaction in QED is known from classical theory of Maxwell et *al*.
There are no classical counterparts for the Strong and Weak interactions.
In general, guess a suitable form of the interaction and confront it with experiment (particle spectrum, known symmetries and conservation laws, cross-sections, decays, ...)

Quite generally, the form of interaction is restricted by requiring
- Lorentz invariance
- locality (the fields evaluated at same space-time point)
- renormalizability

QED is a gauge field theory and renormalizable theories are gauge field theories, i.e. possessing local phase invariance.
Elementary particle physics is almost exclusively concerned with such theories:
QCD and GWS are both gauge filed theories, remarkable generalizations of QED.

Strong interactions – quantum chromodynamics QCD
characterized by an apparently simple Lagrangian, but physical properties very difficult to deduce because of technical problems in formulating perturbation theory and the need of higher order corrections (α_S not so small).

ElectroWeak interactions – GSW model
very complicated Lagrangian, but easy to deal with in perturbation theory.

Gauge Theories: QED and Yang-Mills

U(1) symmetry Lee group

EM U(1) $\phi \to e^{i\alpha}\phi$ but $\partial_\mu \phi \to e^{i\alpha}(\partial_\mu \phi) + \underbrace{i(\partial_\mu \alpha)}\phi$

≠0 if local transformations

EM field and covariant derivative $\partial_\mu \phi + ieA_\mu \phi \to e^{i\alpha}(\partial_\mu \phi + ieA_\mu \phi)$

if $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha$

the EM field keep track of the phase in different points of the space-time

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

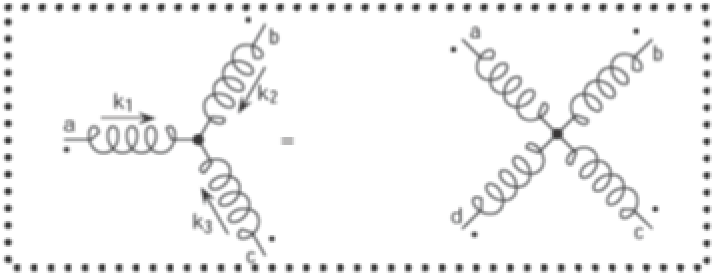
SU(2) and SU(3) symmetry Lee groups

Yang-Mills : non-abelian transformations $\phi \to U\phi$

$\partial_\mu \phi + igA_\mu \phi \to U(\partial_\mu \phi + igA_\mu \phi)$ if $A_\mu \to UA_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{ig[A_\mu, A_\nu]}$$

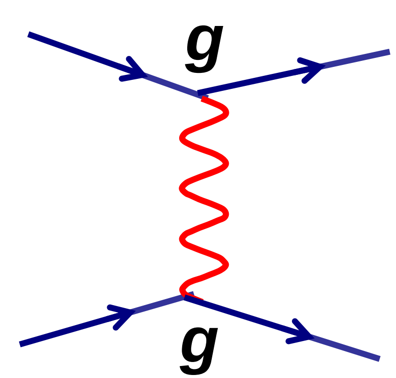
non-abelian int.



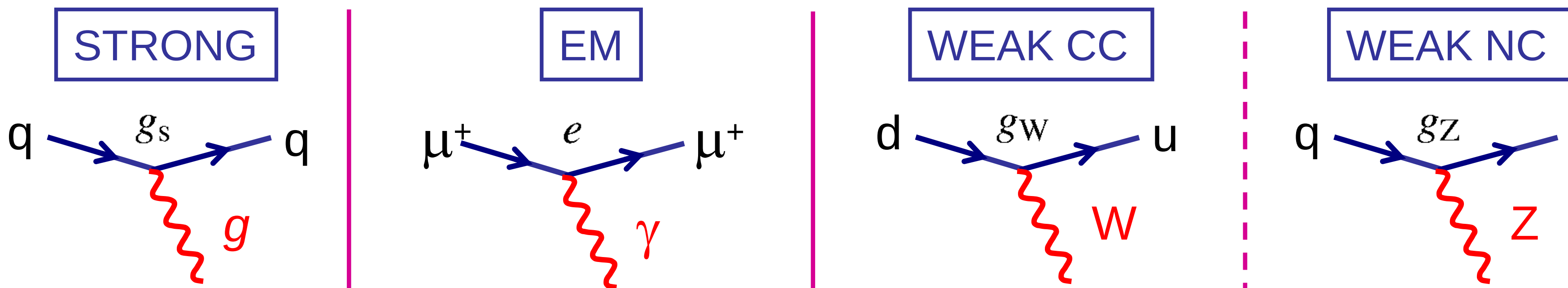
The Mediators

Interactions between fermions are mediated by the exchange of spin-1 Gauge Bosons

Force	Boson(s)	J^P	*m* [GeV]
EM (QED)	Photon γ	1^-	0
Weak	$W^\pm$ / Z	1^-	80 / 91
Strong (QCD)	8 Gluons *g*	1^-	0
Gravity	Graviton?	2^+	0



Interactions of gauge bosons with fermions described by SM vertices



Only quarks
Never changes flavour
$\alpha_s \sim 0.1$

All charged fermions
Never changes flavour
$\alpha \sim 1/137$

All fermions
Always changes flavour

All fermions
Never changes flavour

$\alpha_W \sim 1/40$