Advanced Particle Physics 1

Electromagnetic Interactions

L7 – Electromagnetic Structure of Hadrons

(http://dpnc.unige.ch/~bravar/PPA1/L7)

lecturer

Alessandro Bravar Alessandro.Bravar@unige.ch

tél.: 96210 bureau: EP 206

assistant

Johannes Hulsman Johannes.Hulsman@unige.ch

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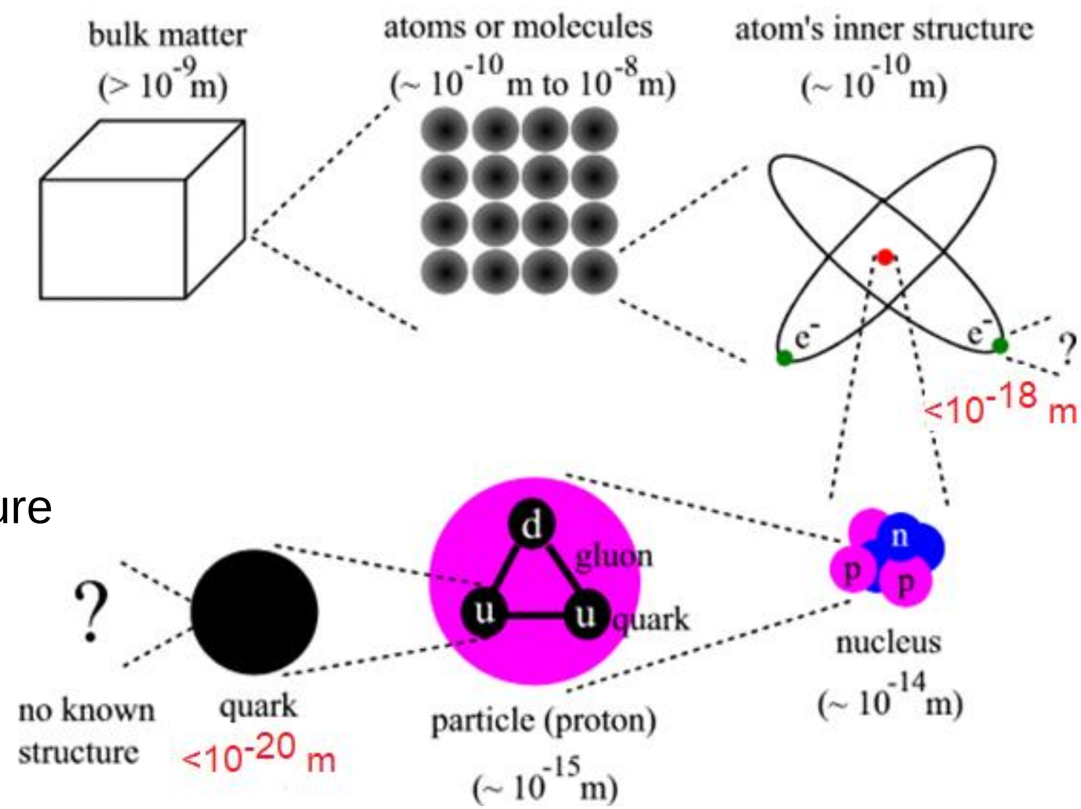
Down Rutherford's path

Scattering of point-like fermions off a structured target (such as a nucleus or nucleon) reveals its internal structure



γ + matter $\Rightarrow$ matter has substructure (atoms and molecules)
distances probed ~ 10^{-10} m

α + atom $\Rightarrow$ atom has structure (electrons + nucleus)
distances probed ~ 10^{-13} m

e + nucleus $\Rightarrow$ nucleus has substructure (protons + neturons)
distances probed ~ 10^{-14} m

e + proton $\Rightarrow$ proton has substructure (partons, i.e. quarks + gluons)
distances probed ~ 10^{-15} – 10^{-18} m

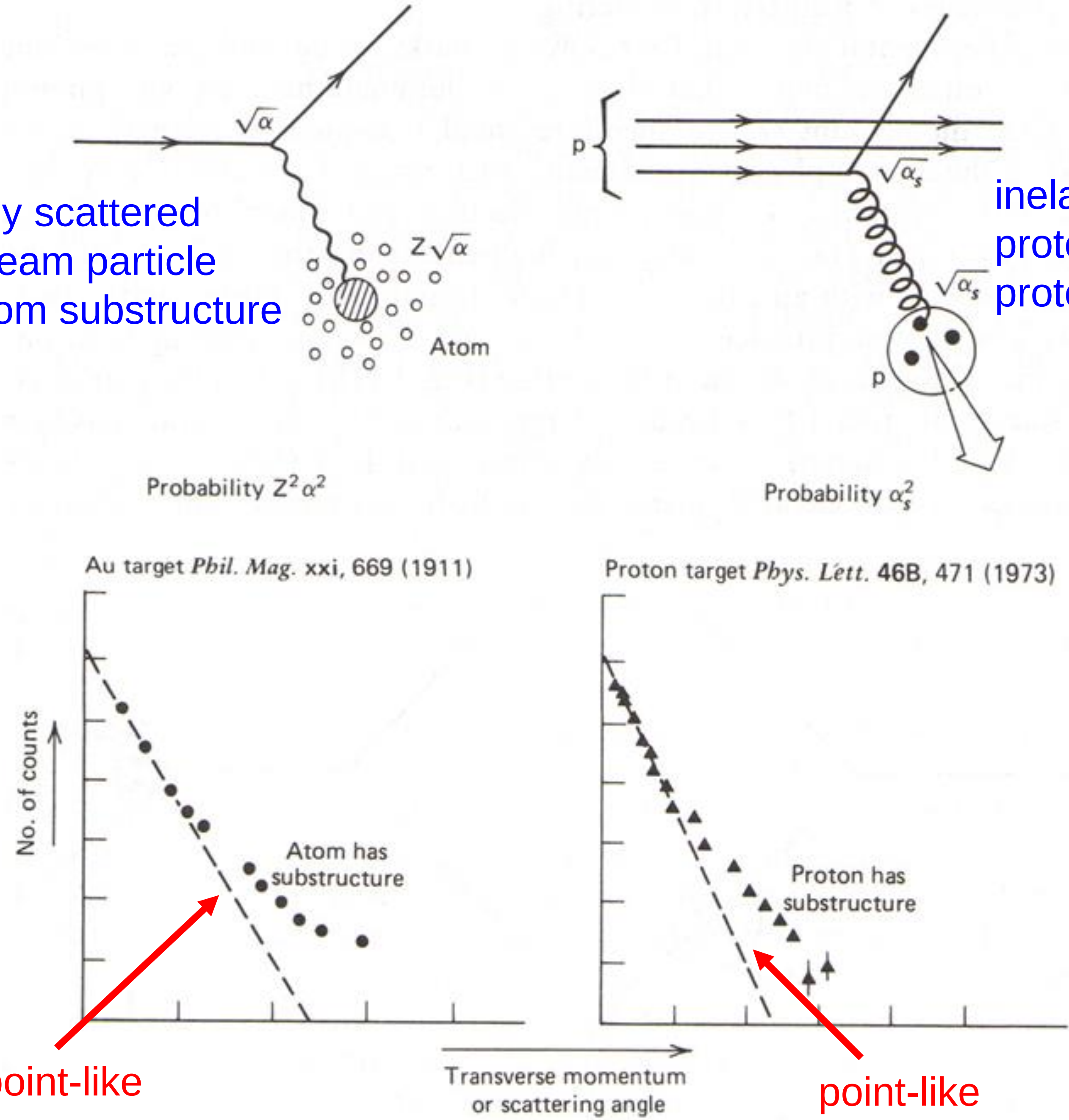
pp $\rightarrow$ jets (qq scattering)
distances probed ~ 10^{-20} m

distance probed $\lambda \approx \dfrac{hc}{Q} \approx \dfrac{200\,\mathrm{MeV \cdot fm}}{Q}$

No evidence for quark nor electron substructure found so far: pointlike down to 10^{-20} m
cfr. electron classical radius $r_e = 2.8 \times 10^{-15}$ m

inelastically scattered charged beam particle reveals atom substructure

inelastically scattered proton beam reveals proton substructure



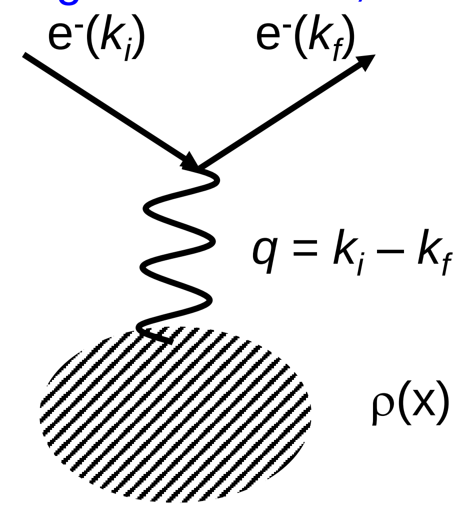
point-like

point-like

Structure – Form Factors

Study the internal structure of a "particle" by analyzing the cross section of the electromagnetic scattering, i.e. observe only the scattered electron.
To be sensitive to the internal structure of a target, the wave length of the probe λ, in this case the virtual photon, must be smaller compared to the charge distribution,

⇒ an incredible amount of information can be extracted from the scattered particle angular and energy distribution



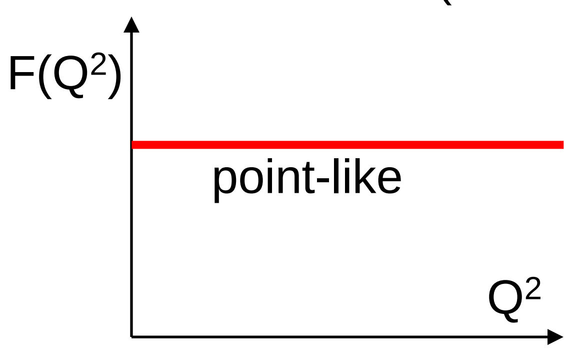
For a cloud of static charge, the angular distribution of the scattered electron results from a convolution of the point-like cross section - Mott scattering - with the charge distribution:

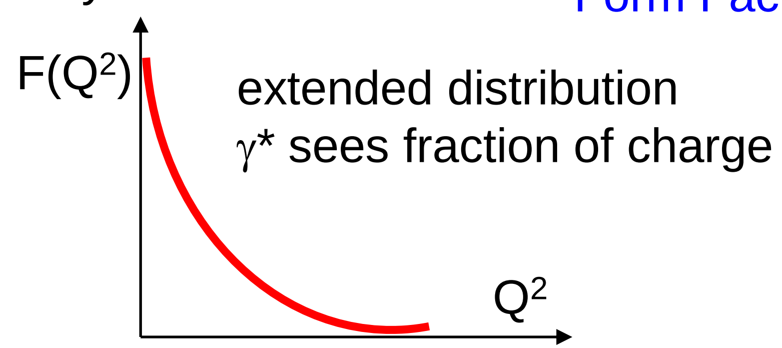
$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left|F\left(q^2\right)\right|^2$$

$$F\left(\vec{q}\right) = \int \rho\left(x\right)\, e^{i\vec{q}\cdot\vec{x}} \mathrm{d}^3 x$$

Form Factor

if we know the force at work (i.e. EM) can study the structure



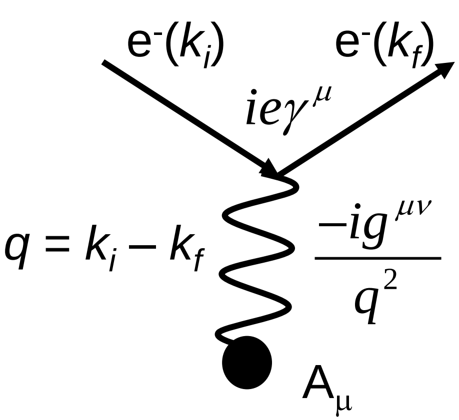


Mott Scattering

The reference process is the
elastic scattering of electrons off a point-like and infinitely heavy target.
We call such process the Mott scattering.
In this process the target does not move nor it is excited or modified, and $E_i = E_f$.

Let's start with the scattering amplitude for a static (no recoil), spinless point-like charge

$$T_{fi} = -i\int \mathrm{d}^4x \; j_\mu^{fi}(x)A^\mu(x) \quad \text{with} \quad j_\mu^{fi} = -e\bar{u}_f\gamma_\mu u_i e^{-iq\cdot x}$$



How to determine $A^\mu(x)$?
The charge current 4-vector of a static, spinless point charge is

$$J^\mu = \left(Ze\delta(x), \vec{0}\right)$$

Since u_f and u_i depend only on momentum,
the integral in T_{fi} leads to the Fourier transform of $A^\mu(x)$:

$$T_{fi} = -i\int \mathrm{d}^4x \; j_\mu^{fi}(x)A^\mu(x) = ie\bar{u}_f\gamma_\mu u_i A^\mu(q) \quad \text{with} \quad A^\mu(q) = \int \mathrm{d}^4x \; e^{-iq\cdot x}A^\mu(x)$$

For a static source, $A^\mu(x)$ does not depend on time and

$$\boxed{A^\mu(q) = \int \mathrm{d}t \; e^{-i(E_i - E_f)t}\int \mathrm{d}^3x \; e^{i\vec{q}\cdot\vec{x}}A^\mu(\vec{x}) = 2\pi\delta\left(E_i - E_f\right)A^\mu(\vec{q})}$$

From Maxwell equations

$$\nabla^2 A^\mu(\vec{x}) = -J^\mu(\vec{x}) \qquad \rightarrow \qquad J^\mu(\vec{q}) = -\int \mathrm{d}^3x \left(\nabla^2 A^\mu(\vec{x})\right) e^{i\vec{q}\cdot\vec{x}}$$

Integrating by parts (two times) gives

$$J^\mu(\vec{q}) = -\int \mathrm{d}^3x \; A^\mu(\vec{x})\left(\nabla^2 e^{i\vec{q}\cdot\vec{x}}\right) = \left|\vec{q}\right|^2 A^\mu(\vec{q}) \qquad \rightarrow \qquad A^\mu(\vec{q}) = \frac{1}{\left|\vec{q}\right|^2} J^\mu(\vec{q})$$

Therefore the scattering amplitude for a time-independent potential is

$$T_{fi} = i2\pi\delta\left(E_i - E_f\right)\left(e\bar{u}_f\gamma_\mu u_i\right)\frac{1}{\left|\vec{q}\right|^2} J^\mu(\vec{q})$$

with the invariant amplitude given by

$$\boxed{-iM_{fi} = ie\bar{u}_f\gamma_\mu u_i \frac{1}{\left|\vec{q}\right|^2} J^\mu(\vec{q})}$$

In the static case, the momentum direction changes, but the energy of the scattered particle is conserved, i.e. $E_i = E_f$, thus $q_0 = 0$ and $q^2 = -|\boldsymbol{q}|^2$

$$-iM_{fi} = \left(ie\bar{u}_f\gamma_\mu u_i\right)\frac{-ig_{\mu\nu}}{q^2}\left(-iJ^\nu(\vec{q})\right)$$

For a static point charge *Ze*, its Fourier transform is $J^0(\vec{q}) = Ze$ and the amplitude becomes

$$\boxed{-iM_{fi} = \left(ie\bar{u}_f\gamma_\mu u_i\right)\frac{-i}{q^2}\left(-iZe\right)}$$

With
$$q^2 = |\vec{q}|^2 = -4k^2 \sin^2(\vartheta/2) \qquad k \equiv |\vec{p}_i| = |\vec{p}_f|$$
where θ is the electron scattering angle
we find a strong angular dependence
$$1/k^4 \sim 1/\sin^4(\vartheta/2)$$
of the Rutherford scattering cross section
(low energy approximation, neglecting the recoil of the target).

For a cloud of static charge ρ(**x**), replace *Ze* with ρ(**x**)
The scattering amplitude for a time-independent charge distribution ρ(**x**) becomes
$$\boxed{T_{fi} = -i2\pi\delta(E_i - E_f)(-e\bar{u}_f\gamma_\mu u_i)\frac{1}{|\vec{q}|^2} ZeF(\vec{q})}$$
with
$$\boxed{F(\vec{q}) = \int \mathrm{d}^3x \ \rho(\vec{x}) \ e^{i\vec{q}\cdot\vec{x}}}$$
the Fourier transform of the normalized spatial charge distribution ρ(**x**).

The Cross Section

The cross section in the laboratory frame is (setting V = 1, the target is fixed)

$$\mathrm{d}\sigma = W_{fi} \times (\text{number of final states}) \frac{1}{(\text{relative incident flux})}$$

$$= \frac{\left|T_{fi}\right|^2}{T} \frac{\mathrm{d}^3 k_f}{(2\pi)^3 2E_f} \frac{1}{v 2E_i} = \frac{\left|T_{fi}\right|^2}{T} \frac{k_f^2 \mathrm{d}k_f \mathrm{d}\Omega}{(2\pi)^3 2E_f} \frac{1}{v 2E_i}$$

Recall $2\pi\delta\left(E_i - E_f\right)^2 = 2\pi\delta\left(E_i - E_f\right)T, \;\; k_f \mathrm{d}k_f = E_f \mathrm{d}E_f, \;\; k_f \approx E_f$

we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi \left(\frac{1}{2} \sum_{\text{spins}} \left| \bar{u}_f \gamma_0 u_i \right|^2 \right) \frac{Z^2 e^4 \left|F(\vec{q})\right|^2}{\left|\vec{q}\right|^4} \frac{1}{(2\pi)^3} \frac{1}{4}$$

The sum over spins ($k/E = \beta = v/c$)

$$\frac{1}{2} \sum_{\text{spins}} \left| \bar{u}_f \gamma_0 u_i \right|^2 = 2\left[k_f^0 k_i^0 + k_f^0 k_i^0 - (k_f k_i - m^2) \right] = 4E^2 \left(1 - \beta^2 \sin^2(\vartheta/2)\right)$$

and finally

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2 \alpha^2 E^2 \left|F(\vec{q})\right|^2}{4k^4 \sin^4(\vartheta/2)} \left(1 - \beta^2 \sin^2(\vartheta/2)\right) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Mott}} \left|F(\vec{q})\right|^2}$$

Summary of Cross Section – Spinless Target

Mott cross section

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \frac{Z^2\alpha^2E^2}{4k^4\sin^4(\vartheta/2)}\left(1-\beta^2\sin^2(\vartheta/2)\right)}$$

non relativistic limit $\beta \to 0$: Rutherford cross section

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Ruth}} = \frac{Z^2e^4}{(4\pi)^2 4E^2\sin^4(\vartheta/2)}}$$

Ultra-relativistic limit $\beta \to 1$

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \frac{Z^2\alpha^2}{4k^2\sin^4(\vartheta/2)}\left(\cos^2(\vartheta/2)\right)}$$

The Form Factor

For relative small momentum transfers (small angle scattering) we can expand the form factor

$$F(\vec{q}) = \int \mathrm{d}^3x\ \rho(\vec{x})\ e^{i\vec{q}\cdot\vec{x}} \approx \int \mathrm{d}^3x\ \left(1 + i\vec{q}\cdot\vec{x} - \frac{1}{2}(\vec{q}\cdot\vec{x})^2 + \cdots\right)\rho(\vec{x})$$

For a spherically symmetric distribution $\rho(\mathbf{x}) = \rho(x)$, $\int \mathrm{d}^3x \left(i\vec{q}\cdot\vec{x}\right)^n \rho(\vec{x}) = 0$ for n odd.
Then

$$F(|\vec{q}|) = 1 - \int \frac{1}{2}\left(|\vec{q}|\, r\cos\vartheta\right)^2 \rho(r) r^2 \mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\cos\vartheta + \cdots$$

$$= 1 - \frac{1}{6}|\vec{q}|^2\, 4\pi \int r^4 \rho(r)\mathrm{d}r + \cdots = 1 - \frac{1}{6}|\vec{q}|^2 \left\langle r^2 \right\rangle + \cdots$$

Low energy scattering measures basically the quadratic mean radius $\sqrt{\left\langle r^2 \right\rangle}$ of the target.

If the charge distribution is exponential,
$\rho(x) \sim \exp(-\Lambda r)$,
we obtain the so called dipole form factor

$$F(|\vec{q}|) \sim \left(1 + \frac{|\vec{q}|^2}{\Lambda^2}\right)^{-2}$$

some Fourier transforms

$$\delta(r) \qquad \Rightarrow F\left(q^2\right) = 1$$

$$\rho(r) \sim \rho_0 e^{-\Lambda r} \Rightarrow F\left(q^2\right) \sim \left(1 + \frac{q^2}{\Lambda^2}\right)^{-2}$$

$$\rho(r) \sim \rho_0 \frac{e^{-\Lambda r}}{r} \Rightarrow F\left(q^2\right) \sim \left(1 + \frac{q^2}{\Lambda^2}\right)^{-1}$$

$$\rho(r) \sim \rho_0 e^{-(\mathrm{M}r)^2} \Rightarrow F\left(q^2\right) \sim e^{-q^2/4\mathrm{M}^2}$$

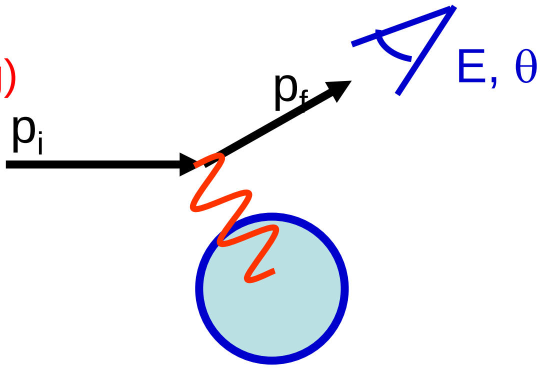
Electromagnetic Structure of Hadrons

Hadrons are composite objects and have structure:

magnetic moment ≠ point-like Dirac particle (mid ´30s)
proliferation of hadrons (50’ …)
elastic electron – Nucleon scattering (low energy !) (mid ´50s)
quark model (p = |uud>, π+ =|u$\bar{d}$>) (´60s)
deep inelastic scattering (´70s …)

protons are stable, proton targets OK
neutrons are unstable (decay after ~15 min) → no neutron targets, use D and ^{3}He instead
(though nuclear model corrections are required …)

Observe only scattered electron (elastic scattering, inclusive scattering)
⇒ incredible amount of information can be extracted from the scattered electron angular and energy distribution



Note that this distribution depends on the force and structure
⇒ if we know the force at work (i.e. γ probe) can study the structure

Elastic scattering: explore global properties of charge (and magnetization) and current dist.
Inelastic scattering: probe internal structure

One photon exchange approximation (Born approx.):
Is one photon exchange good enough (α = 1/137) ? to 1%.
Probability of exchanging 2 or more photos is very small (< 1%).
For Q^2 < 1000 GeV^2 Z^0 (EW) effects are negligible.

e – Nucleus Scattering

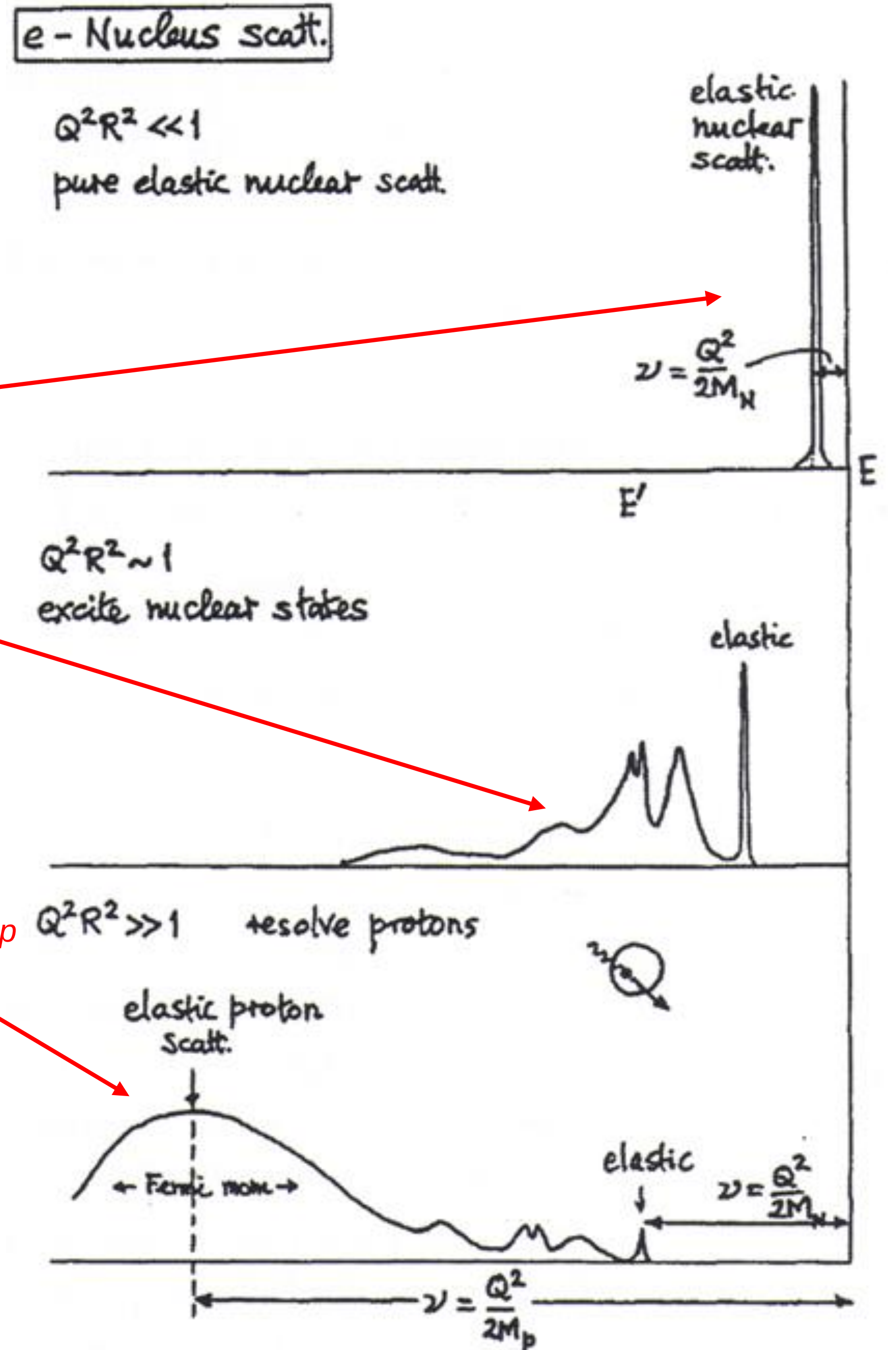
The effective probe is the exchanged virtual photon of wave length $\lambda\!\!\!^{-} = \hbar c / Q$.
($Q^2 = -q^2$)

For low Q^2 (~0.01 GeV^2) the nucleus tends to recoil as a whole with $\nu = Q^2 / 2M_A$.
($\nu = E_i - E_f$)

As Q^2 increases nuclear states are excited.

At higher Q^2 (~0.1 GeV^2)
the nuclear structure is resolved,
and the electron scatters elastically off the constituent nucleons (protons) with $\nu = Q^2/2M_p$
(broad distribution due to Fermi motion).

At even higher Q^2 (> 0.1 GeV^2) the nucleon structure is revealed
(first nucleon resonances are excited).



e – Nucleon Scattering

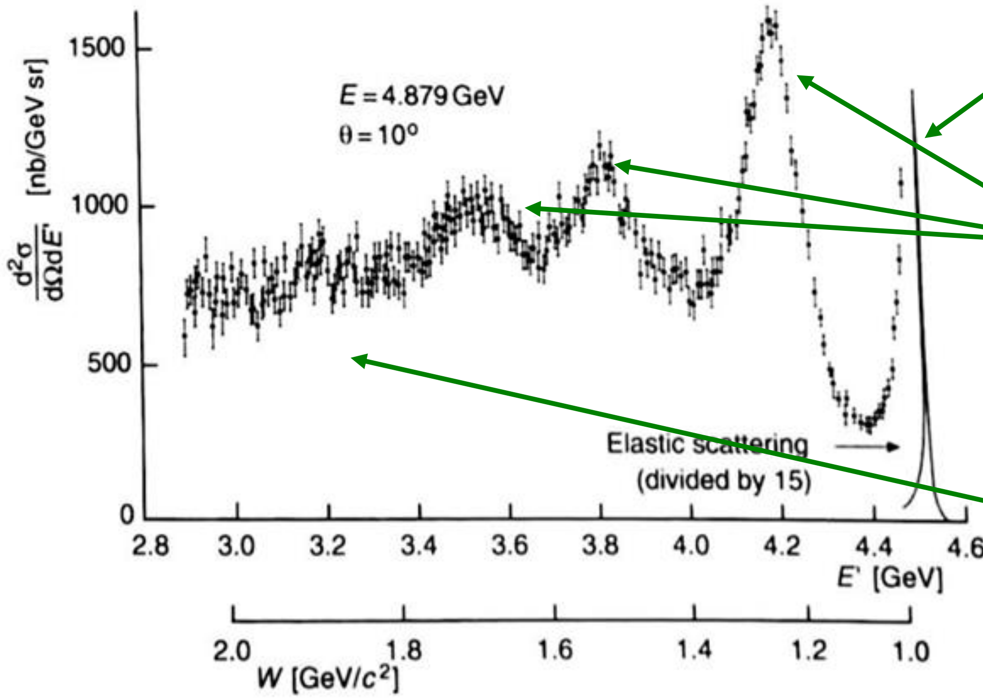
Scattering of 4.879 GeV electrons from protons at rest

Place detector at 10^o to beam and measure the energy of scattered e^-

Kinematics fully determined from the electron energy and angle!

For this energy and angle determine the invariant mass W of the final state hadronic sys

$$W^2 = (p+q)^2 = M_p^2 + 2M_p\nu + q^2$$



Elastic scattering

proton remains intact

$W = M$

Inelastic (resonance) scattering

produce "excited states"

of the proton e.g. $\Delta^+(1232)$

$W = M_\Delta$

Deep Inelastic Scattering

proton breaks up resulting

in a many particle final state

DIS = large W

Due to the proton internal structure, elastic scattering at high Q^2 is unlikely and inelastic reactions, where the proton breaks up dominate.

For inelastic scattering the mass of the final state hadronic system $W = M_X > M_p$.

Elastic Electron – Proton Scattering

We cannot apply directly these results to a nucleon target because
- the spin (magnetic moment) of the nucleon contributes (nucleons have spin ½)
- the nucleon does not stay at rest but it recoils.

If the proton were point-like, with charge +e, mass *M* and magnetic moment *e/2M*, we could use the result of the $e^-\mu^-$ scattering at high energy (m_e~0, $m_\mu = M$)

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{\text{LAB}} = \frac{\alpha^2}{4E^2\sin^4(\vartheta/2)}\frac{E'}{E}\left(\cos^2(\vartheta/2) - \frac{q^2}{2M^2}\sin^2(\vartheta/2)\right)}$$

where E' is the energy of the scattered electron (recoil of the target)

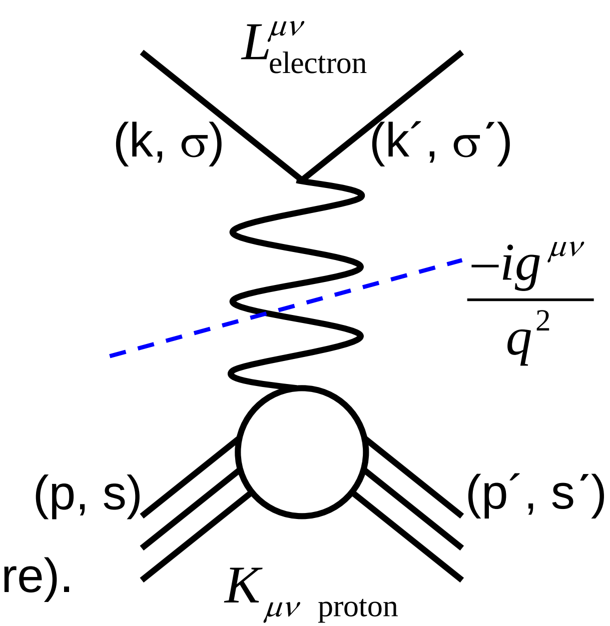
$$E' = E/\left(1 + (2E/M)\sin^2(\vartheta/2)\right)$$

But the proton is not a simple point charge!

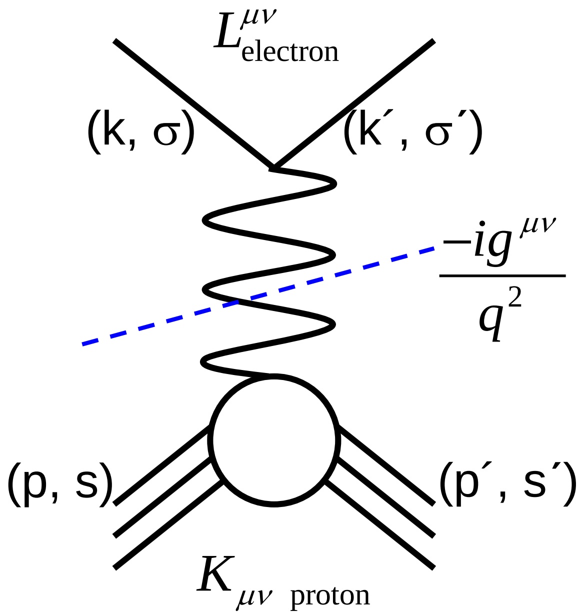
The essential point is that the electron vertex and the photon propagator are unchanged, so we can still write

$$\boxed{\left\langle |M|^2 \right\rangle = \frac{e^4}{q^4} L^{\mu\nu}_{\text{electron}} K_{\mu\nu\ \text{proton}}}$$

i.e. factor the cross section, with $L_{\mu\nu}$ the leptonic tensor an $K_{\mu\nu}$ a tensor describing the proton vertex (proton structure).



elastic scattering: the nucleon remains in its ground state
γ^* absorbed by the nucleon as a whole
The cross section depends on a single variable q^2.



$\Rightarrow$ vertex function modified by the proton structure (μ_p)
we follow the same procedure as for $e^-\mu^-$ scattering,
however we need to construct a more general form
for the hadron vertex Γ^μ and current (Lorentz 4-vector !)
with the quantities at our disposal
i.e. p, p', and the γ matrices
(the operator γ^μ alone cannot be use to describe
the hadron current):

$$j^\nu_{el} = \bar{u}(k',\sigma')\left(-ie\gamma^\nu\right)u(k,\sigma)$$

$$J^\mu_{proton} = \bar{u}(p',s')\left(ie\Gamma^\mu\right)u(p,s)$$

The most general form for Γ^μ is (Lorentz 4-vector)

$$\boxed{\Gamma^\mu = \gamma^\mu K_1 + i\sigma^{\mu\nu}(p'-p)_\nu K_2 + i\sigma^{\mu\nu}(p'+p)_\nu K_3 + (p'-p)^\mu K_4 + (p'+p)^\mu K_5}$$

where the factors $K_i = K_i(q^2)$ describe the proton form (though not all independent!)
and depend on a single variable q^2.
There are no terms with γ^5, since γ^* exchange conserves parity
(not true for polarized scattering or weak interactions).

With the help of the Gordon decomposition we can eliminate one of the combinations, for example (p' + p)

$$(p'+p)^{\mu} = 2M\gamma^{\mu} - i\sigma^{\mu\nu}(p'-p)_{\nu}$$

and we are left with three independent form factors.
Rearranging the terms ($q = p' - p$)

$$\boxed{\Gamma^{\mu} = \gamma^{\mu}F_1(q^2) + \frac{i\kappa}{2M}\sigma^{\mu\nu}q_{\nu}F_2(q^2) + q^{\mu}F_3(q^2)}$$

where we have introduced a new set of factors $F_i = F_i(q^2)$ (combinations of $K_j(q^2)$).
We call κ the anomalous magnetic moment of the proton.

Next, apply the current conservation ($q_{\mu}J^{\mu} = 0$ or $\delta_{\mu}J^{\mu} = 0$), from which follows $F_3(q^2) = 0$.

The most general form for the hadron current is finally

$$\boxed{J^{\mu} = -e\,\bar{u}(p',s')\Gamma^{\mu}u(p,s) = -e\,\bar{u}(p',s')\left[\gamma^{\mu}F_1\left(q^2\right) + \frac{i\kappa}{2M}\sigma^{\mu\nu}q_{\nu}F_2\left(q^2\right)\right]u(p,s)}$$

$F_1(q^2)$ and $F_2(q^2)$ are called the proton Dirac and Pauli form factors.
At small $-q^2$ the internal structure of the proton does not appear, because the photon cannot resolve the details of the charge distribution.
In this limit, the proton appears effectively as a point-like fermion with charge +e and total magnetic moment $\mu_p = (1 + \kappa)e/2M$.
It follows that for the proton $F_1^p(0) = 1$ and $F_2^p(0) = 1$.
Experimentally we find the proton anomalous magnetic moment $\mu_p = 1.79$.

Rosenbluth Formula

To derive the cross section for ep → ep scattering follow the same procedure as for $e\mu$ (calculate lepton and hadron tensors and contract them).

(def $\tau = Q^2 / 4M^2$)

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{LAB}} = \left(\frac{\alpha^2}{4E^2 \sin^4(\vartheta/2)}\right)\frac{E'}{E}\left\{\left(F_1^2 + \tau\kappa^2 F_2^2\right)\cos^2(\vartheta/2) + 2\tau\left(F_1 + \kappa F_2\right)^2 \sin^2(\vartheta/2)\right\}}$$

This parametrization in terms of the form factors is known as the Rosenbluth formula ('50) It reduces to what we have found for the EM scattering in the limit of point-like fermions, with $\kappa = 0$, $F_1(q^2) = 1$ for all q2.

The Rosenbluth formula contains a cross term $F_1 \cdot F_2$, which is inconvenient for interpreting the measurements.
Define a linear combination of F_1 and F_2 (Sachs Form Factors)

G_E – electric form factor
G_M – magnetic form factor

$$G_E = F_1 - \tau\kappa F_2 \qquad G_M = F_1 + \kappa F_2$$

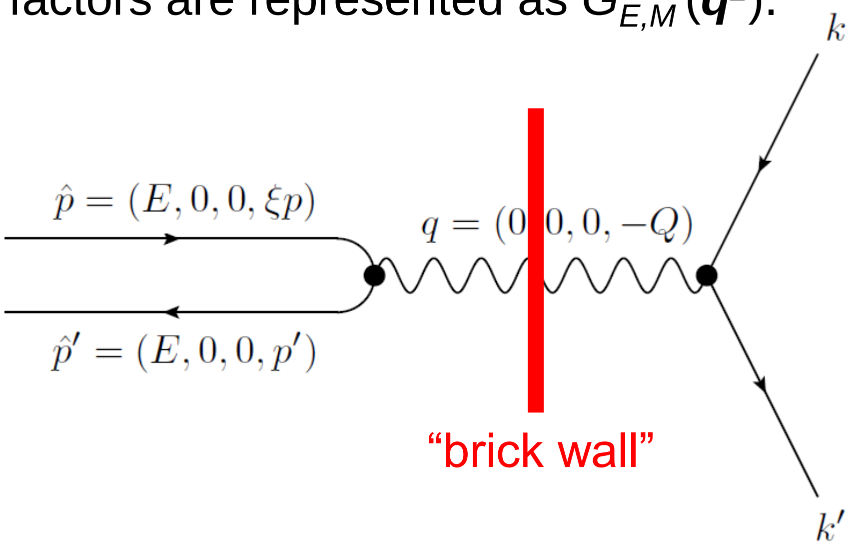
then

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{LAB}} = \left(\frac{\alpha^2}{4E^2 \sin^4(\vartheta/2)}\right)\frac{E'}{E}\left\{\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2(\vartheta/2) + 2\tau G_M^2 \sin^2(\vartheta/2)\right\}}$$

At low $Q^2 \to 0$ $G_E(0) = 1$ and $G_M(0) = 1 + \kappa$.

The Breit Frame

We would like to interpret the Fourier transforms of G_E and G_M as the spatial charge and magnetic moment distributions of the proton. Because of the recoil of the proton (E'/E) this is not possible except in a special reference system, the Breit frame, defined by $\boldsymbol{p}' = -\boldsymbol{p}$. In this reference system, the proton is reflected with no energy transfer, like a ideal elastic ball on a brick wall. For each Q^2, there is a Breit frame in which the form factors are represented as $G_{E,M}(\boldsymbol{q}^2)$.



$\vec{k} = +\vec{q}/2$ $\quad$ $\vec{k}' = -\vec{q}/2$ $\quad$ electron

$\vec{p} = -\vec{q}/2$ $\quad$ $\vec{p}' = +\vec{q}/2$ $\quad$ proton

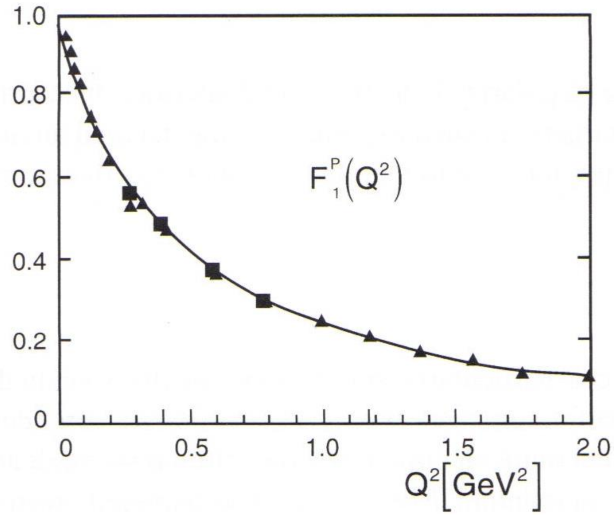
$q^\mu = (0, \vec{q})$ $\quad$ $q^0 = \nu = 0$ $\quad$ photon

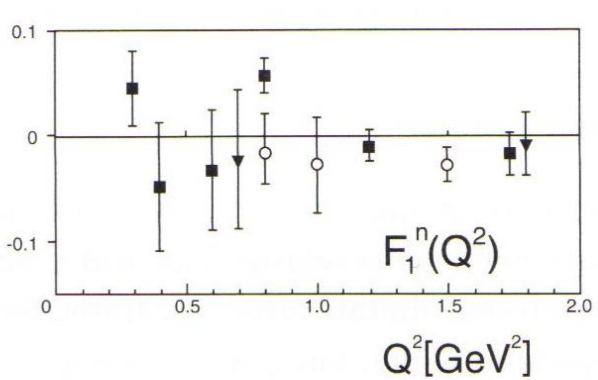
Since $q^2 < 0$ we can boost the photon along its direction of propagation such that q^0 vanish, i.e. $\nu = 0$: the photon carries momentum $\boldsymbol{q}$ but no energy
$\Rightarrow$ no energy transfer to the proton: $E_i = E_f$!

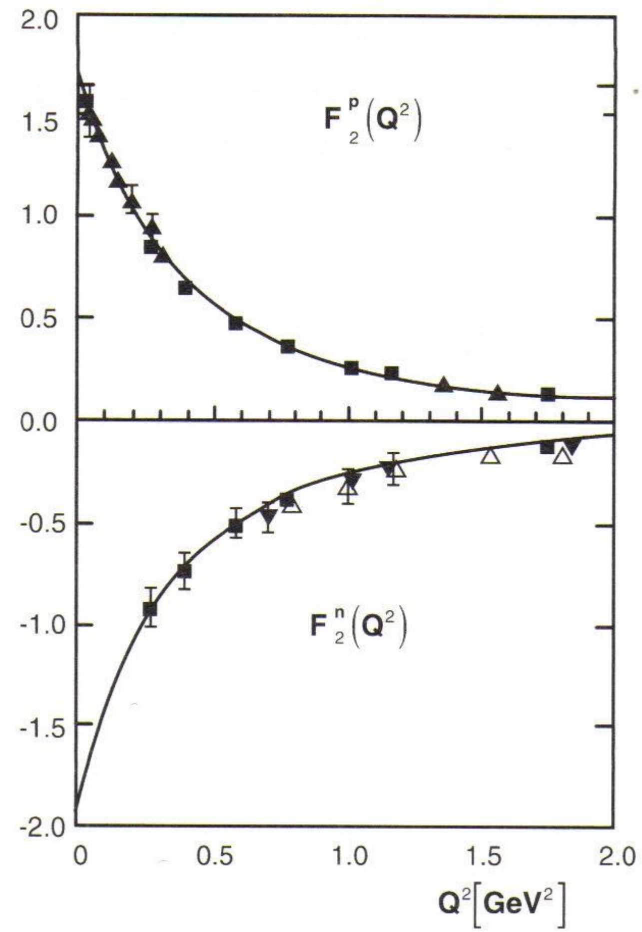
The Breit frame is also know also as the infinite momentum frame, since the proton moves with very high momentum toward the photon.

F_1 and F_2 Elastic Form Factors

The Form Factors are measured by varying Q^2
i.e. by changing the beam energy E and/or by measuring at different scattering angles θ.



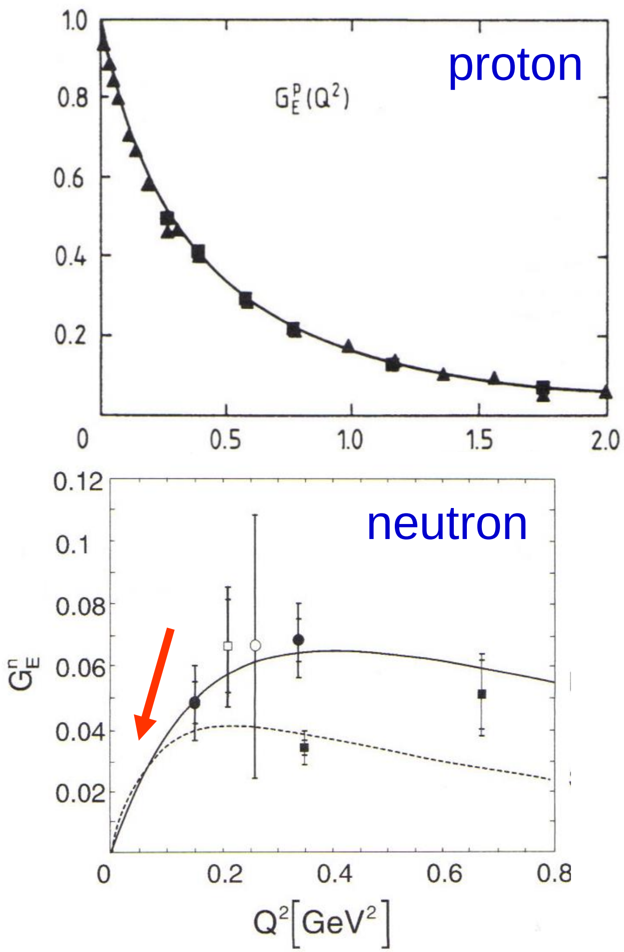


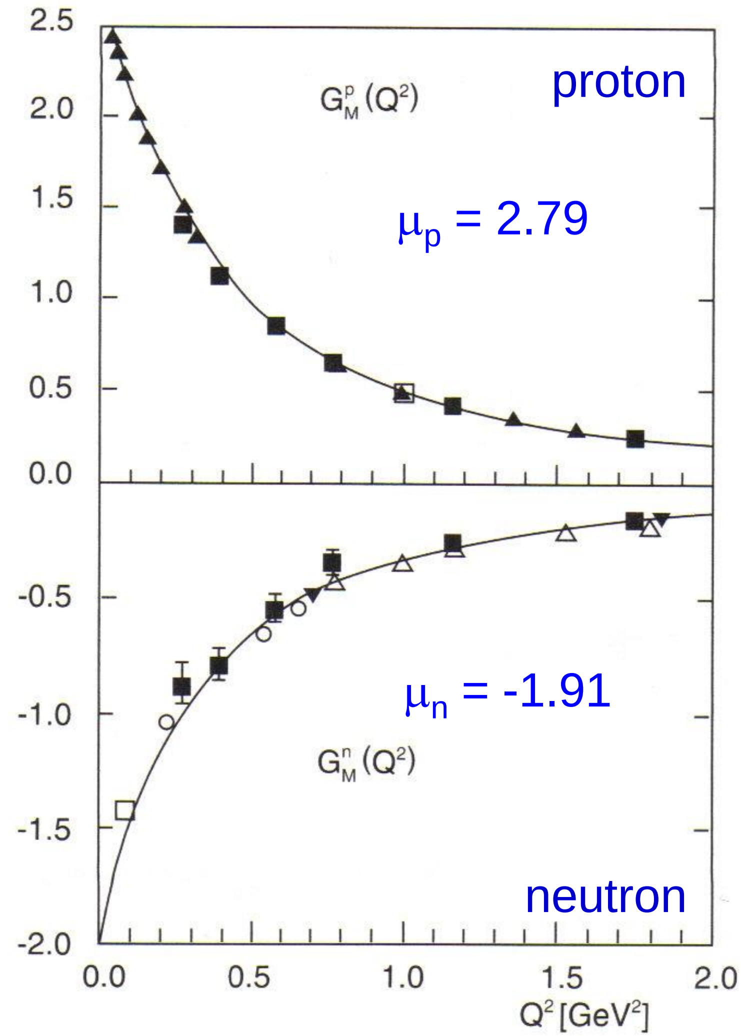


the sign of the proton and neutron magnetic moment is opposite

G_E and G_M Elastic Form Factors

The Form Factors are measured by varying Q^2
i.e. by changing the beam energy E and/or by measuring at different scattering angles θ.



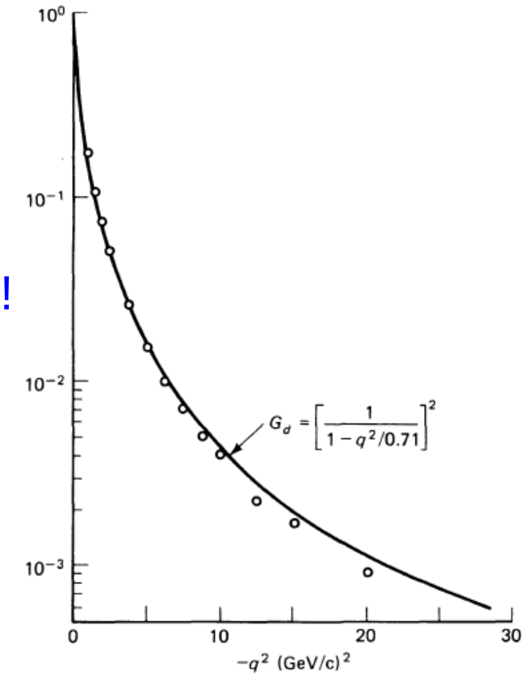


$G_E^n \neq 0$ interpreted as if the neutron has a + core surrounded by a – cloud

The Dipole Form

The global behavior of the proton form factors over a wide range of q^2 is quite well represented by a dipole form

$$G \sim \left(1 - \frac{q^2}{\Lambda^2}\right)^{-2}$$



This indicates an exponential spatial distribution.
Pure phenomenological observation, no strong theoretical basis!

The slopes of the form factors $G_{E,M}$ at $Q^2 = 0$ measure the mean square radii of the charge and magnetization distributions

$$\langle r^2 \rangle = -6 \left.\frac{dG(Q^2)}{dQ^2}\right|_{Q2=0}$$

with the following empirical values

$$\sqrt{\langle r_E{}^2 \rangle}_p = (0.86 \pm 0.01) \;\; \text{fm} \qquad \sqrt{\langle r_M{}^2 \rangle}_p = (0.86 \pm 0.06) \;\; \text{fm}$$

$$\sqrt{\langle r_M{}^2 \rangle}_n = (0.89 \pm 0.07) \;\; \text{fm}$$

Nucleons have a size of about 1 fm!

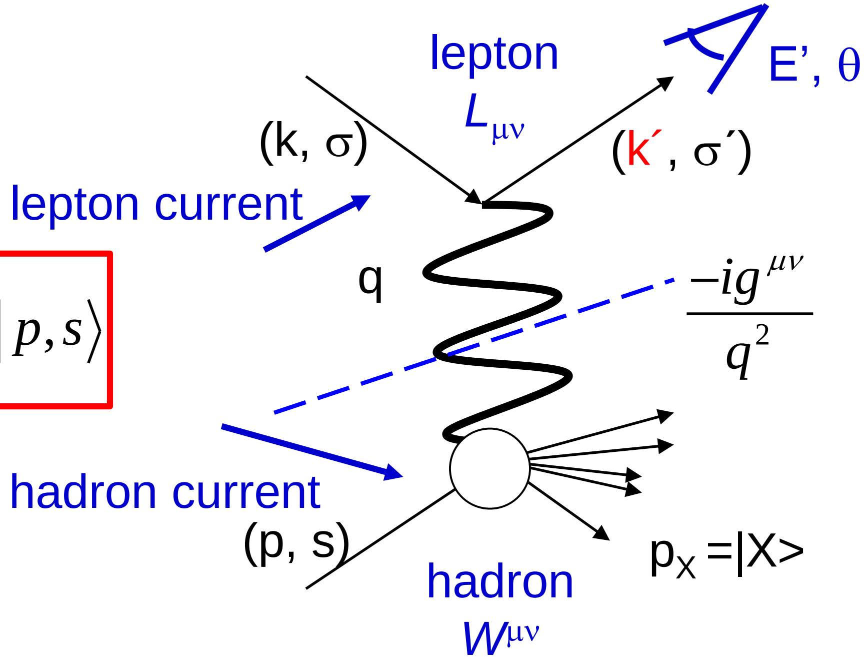
Electron – Nucleon Inelastic Cross Section

Assume a general final hadronic state |X>

e + p → e´ + hadrons (unobserved)

The invariant amplitude can still be factorized
In a leptonic and hadronic current.

$$-iM = \bar{u}(k',\sigma')\left(-ie\gamma^{\mu}\right)u(k,\sigma)\frac{-ig_{\mu\nu}}{q^2}\left\langle X\left|J^{\nu}_{had}\right|p,s\right\rangle$$



Follow the procedure used for calculating
e + μ → e + μ scattering
(separate sums over lepton and hadron spins)

$$d\sigma \sim \frac{1}{Q^4} L^{el}_{\mu\nu} W^{\mu\nu}_{had}$$

mass of hadronic system

$$W^2 = (p+q)^2 = M_p^2 + 2M_p\nu + q^2$$

evaluate $|M|^2$ and sum over all possible hadronic states
unpolarized electrons
no final state polarization observed

$$d\sigma = \frac{e^4}{Q^4}\frac{d^3k'}{(2\pi)^3 2E'}\frac{4\pi M}{4ME} L_{\mu\nu} W^{\mu\nu}(q,p)$$

Hadron Tensor

The hadron tensor $W^{\mu\nu}$ parameterizes our ignorance of the hadron structure at the other end of the photon propagator

- average over initial proton spin states (Σ_{spin})
- sum over all hadronic final states |X> (and spins)
- integrate over all final state particle momenta (only particles on mass shell can be observed!)
- Q^2 and ν (or p and q) are independent

$$\boxed{W_{\mu\nu}(p,q) = \frac{1}{4\pi M}\frac{1}{2}\sum_{spins}\sum_X \langle p,s|J^\dagger{}_\mu(0)|X\rangle\langle X|J(0)_\nu|p,s\rangle(2\pi)^4\delta^{(4)}(p+q-p_X)}$$

The lepton tensor $L^{\mu\nu}$ is the same as for eμ scattering (symmetric in $\mu\leftrightarrow\nu$)

$$L^{\mu\nu}_{el} = 2\left[k'^\mu k^\nu + k^\mu k'^\nu - (k\cdot k')g^{\mu\nu}\right]$$

The most general form for $W^{\mu\nu}$ constructed out of $g^{\mu\nu}$ and independent momenta p and q (γ^μ matrices are not included since we have already averaged over spins)

$$\boxed{\begin{aligned} W^{\mu\nu} &= -W_1 g^{\mu\nu} + \frac{W_2}{M^2}p^\mu p^\nu - i\frac{W_3}{2M^2}\varepsilon^{\mu\nu\sigma\tau}p_\sigma q_\tau \\ &\quad + \frac{W_4}{M^2}q^\mu q^\nu + \frac{W_5}{M^2}(p^\mu q^\nu + q^\mu p^\nu) + i\frac{W_6}{2M^2}(p^\mu q^\nu - q^\mu p^\nu) \end{aligned}}$$

$W_i = W_i(\nu, Q^2)$ or $W_i = W_i(p, q)$ – proton structure functions

(W_3, W_6 reserved for parity-violating structure functions in ν scattering, γ replaced by $W^\pm$)

The Inelastic Cross Section

The hadronic tensor $W^{\mu\nu}$ can be simplified (not all W_i are independent), noting

parity invariance → symmetric form (not for ν scattering) $W_i^{\mu\nu} = W_i^{\nu\mu} \quad \left(W_3, W_6 \to 0\right)$

current conservation $q_\mu W_i^{\mu\nu} = q_\nu W_i^{\nu\mu} = 0$

implies

$$W_4 = \left(M^2 / q^2\right) W_1 + \left(p \cdot q / q^2\right)^2 W_2 \qquad W_5 = -\left(p \cdot q / q^2\right) W_2$$

Finally

$$\boxed{W^{\mu\nu} = W_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2}{M^2}\left(p^\mu - \frac{p \cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p \cdot q}{q^2} q^\nu\right)}$$

Unlike for elastic scattering, there are two independent variables, e.g. ν and q^2 .

Contract $L_{\mu\nu}$ and $W^{\mu\nu}$

$$\boxed{L_{\mu\nu} W^{\mu\nu} = 4 W_1 (k \cdot k') + \frac{2 W_2}{M^2}\left[2 (p \cdot k)(p \cdot k') - M^2 k \cdot k'\right]}$$

In lab frame we have $p = (M, \mathbf{0})$:

$$(p \cdot k)(p \cdot k') = M^2 E E' \quad \text{and} \quad (k \cdot k') = E E' (1 - \cos\vartheta) = 2 E E' \sin^2(\vartheta / 2)$$

so finally

$$\boxed{L_{\mu\nu} W^{\mu\nu} = 4 E E' \left\{W_2(\nu, Q^2) \cos^2(\vartheta / 2) + 2 W_1(\nu, Q^2) \sin^2(\vartheta / 2)\right\}}$$

The flux and phase space factors are

$$F = 4\left[(k\cdot p)^2 - m^2 M^2\right]^{1/2} = 4EM$$

$$\mathrm{d}Q = \frac{\#\ \text{final states}}{\text{particles}} = \frac{\mathrm{d}^3 k'}{(2\pi)^3 2E'}$$

The invariant amplitude for e + p → e + X is given by
(replace the muon tensor of e + μ → e + μ with the hadronic tensor) :

$$\boxed{\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{e^4}{q^4} 4\pi M\ L_{\mu\nu} W^{\mu\nu}(q,p)}$$

where $4\pi M$ is a normalization factor for $W_{\mu\nu}$.
The integration of the phase space of the hadronic system has been absorbed into $W_{\mu\nu}$!

and $$\boxed{\mathrm{d}\sigma = \frac{\left\langle \left|M_{fi}\right|^2 \right\rangle}{F}\mathrm{d}Q = \frac{1}{4\left[(k\cdot p)^2 - m^2 M^2\right]^{1/2}} \left\{ \frac{e^4}{q^4} 4\pi M\ L_{\mu\nu} W^{\mu\nu}(q,p) \right\} \frac{\mathrm{d}^3 k'}{(2\pi)^3 2E'}}$$

Finally, we obtain the angular distribution for inelastic e + p → e + X scattering

$$\boxed{\left.\frac{\mathrm{d}\sigma}{\mathrm{d}E'\mathrm{d}\Omega}\right|_{\mathrm{LAB}} = \frac{\alpha^2}{4E^2 \sin^4(\vartheta/2)} \left\{ W_2(\nu, Q^2)\cos^2(\vartheta/2) + 2W_1(\nu, Q^2)\sin^2(\vartheta/2) \right\}}$$

Summary of Cross Section Formulae

The structure of the target becomes apparent if we summarize the various formulae. For all the reactions the differential cross section can be written in the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \frac{4\alpha^2 E'^2}{q^4}\{\ldots\}$$

with

eμ → eμ $$\{\ \}_{e\mu\to e\mu} = \left(\cos^2\frac{\vartheta}{2} - \frac{q^2}{2m^2}\sin^2\frac{\vartheta}{2}\right)\delta\left(\nu + \frac{q^2}{2m}\right)$$

elastic *ep* scattering $$\{\ \}_{ep\to ep} = \left(\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\frac{\vartheta}{2} + 2\tau G_M^2\sin^2\frac{\vartheta}{2}\right)\delta\left(\nu + \frac{q^2}{2M}\right)$$

inelastic *ep* scattering $$\{\ \}_{ep\to eX} = \left(W_2\left(\nu, Q^2\right)\cos^2\frac{\vartheta}{2} + 2W_1\left(\nu, Q^2\right)\sin^2\frac{\vartheta}{2}\right)$$

The cross sections for elastic scattering can be integrating over E' (δ functions) with the result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E^2\sin^4(\vartheta/2)}\frac{E'}{E}\{\ldots\}$$

and the Mott cross section (for a static target E' / E = 1)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4(\vartheta/2)}\frac{E'}{E}\cos^2(\vartheta/2)$$