Advanced Particle Physics 2

Strong Interactions and Weak Interactions

L2 – The Quark Model

(http://dpnc.unige.ch/~bravar/PPA2/L2)

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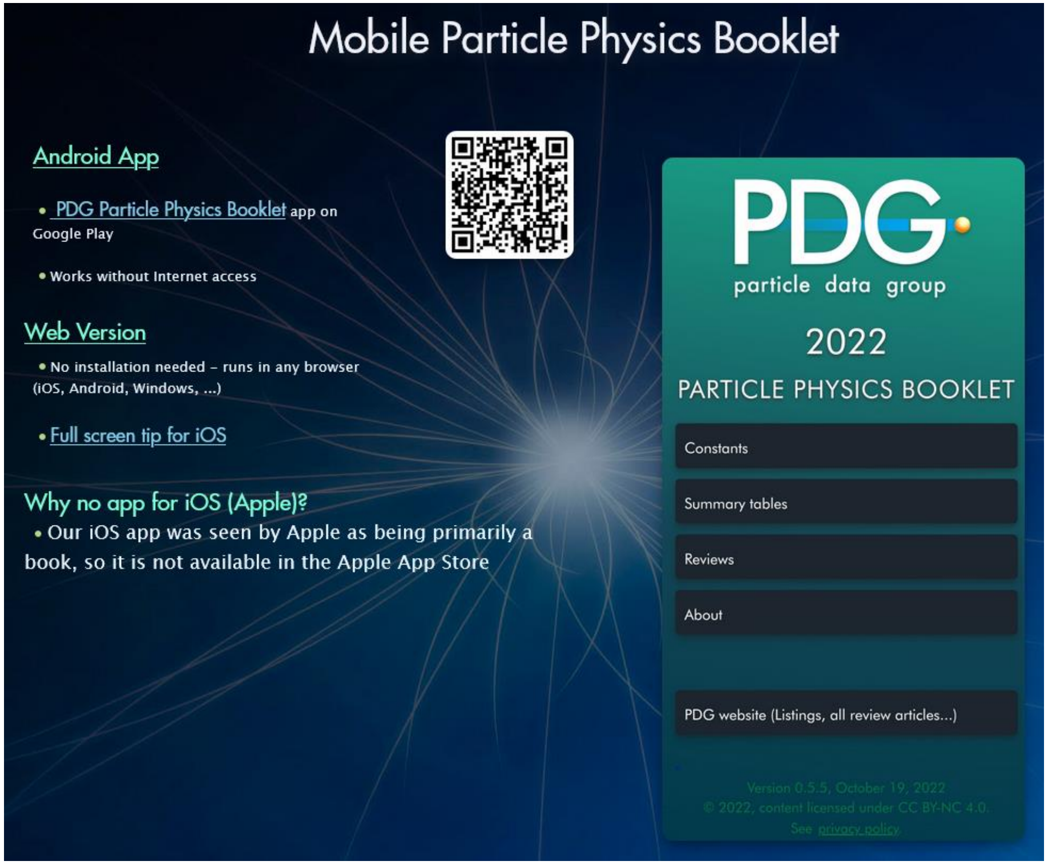
Particle Physics Booklet - PDG

https://pdg.lbl.gov/

lists all known particles
summarizes important physics
summarizes the detectors
can be even download to your phone



Mobile Particle Physics Booklet

Android App

- PDG Particle Physics Booklet app on Google Play
- Works without Internet access

Web Version

- No installation needed – runs in any browser (iOS, Android, Windows, ...)
- Full screen tip for iOS

Why no app for iOS (Apple)?

- Our iOS app was seen by Apple as being primarily a book, so it is not available in the Apple App Store

Version 0.5.5, October 19, 2022
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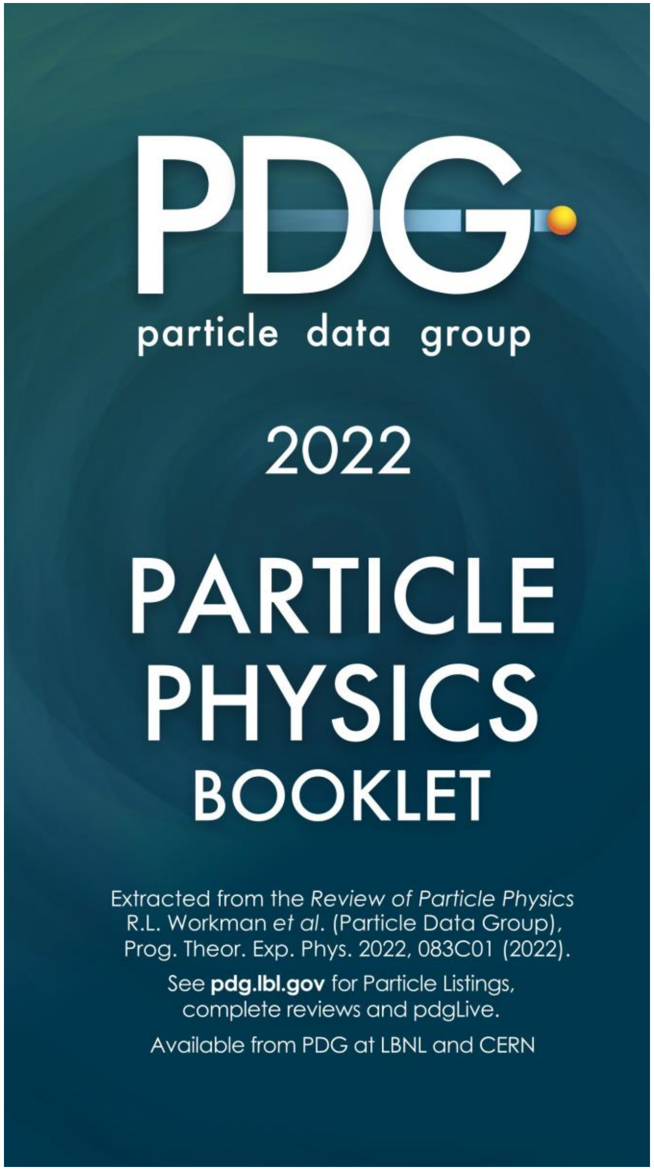


PDG
particle data group

2022

PARTICLE
PHYSICS
BOOKLET

Extracted from the *Review of Particle Physics*
R.L. Workman *et al.* (Particle Data Group),
Prog. Theor. Exp. Phys. 2022, 083C01 (2022).

See **pdg.lbl.gov** for Particle Listings,
complete reviews and pdgLive.

Available from PDG at LBNL and CERN

There is also the full version (~2270 pages)

From Hadrons to Quarks

1932 Discovery of the neutron
(beginning of flavor physics → isospin)

1935


1935 Yukawa postulates the existence of the pion
(strong interaction theory)

1949


1947 Discovery of the pion (initially confused with the muon)

1950


’50s Discovery of hadronic resonances
Discovery of strange particles
Proliferation of hadrons
e-p elastic scattering (→ the proton is not pointlike)

1961


1964 Introduction of quarks

1969


1973 e-p deep inelastic scattering
(→ the proton is made of pointlike constituents - partons)

1990


1973 “November revolution”: Discovery of charm (J / Ψ)

1976


1973 QCD and Asymptotic freedom

2004


1977 Discovery of the bottom quark (Υ)

1995 Discovery of the top quark

Properties of Particles

What can we measure ?

- mass
- lifetime
- decay modes and branching ratios
- magnetic moment
- (internal) quantum numbers
  - spin
  - flavor (quark content)
  - parity transformation
  - charge conjugation transformation

Interactions

Isospin

Observations

$m_p = 938.272$ MeV/c^2 $m_n = 939.565$ MeV/c^2

and

$V_{pp} \approx V_{pn} \approx V_{nn}$ charge independence of the strong force

In 1932 Heisenberg proposed that, if one could switch off the electric charge, protons and neutrons would be indistinguishable (as far as the strong force is concerned). Think of an electron in a magnetic field: if one switches off the magnetic field, the two spin states of the electron are indistinguishable.

The proton and the neutron are two manifestations of one and same particle: the nucleon

The nucleon may be viewed as having an internal d.o.f. with 2 allowed states, the proton and the neutron, which the nuclear force does not distinguish.

The new flavour symmetry of the strong interaction – isospin – has the same transformation properties as SPIN!

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Each nucleon has isospin $I = ½$,
with $I_3 = +½$ for protons and $I_3 = -½$ for neutrons.
The mathematics is a carbon copy of spin → isospin (SU(2) algebra).

The observed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantities, i.e. the conservation of the isospin I and I_3.

Strong interactions conserve I and I_3

Electromagnetic interactions conserve only I_3 (i.e. the electric charge)

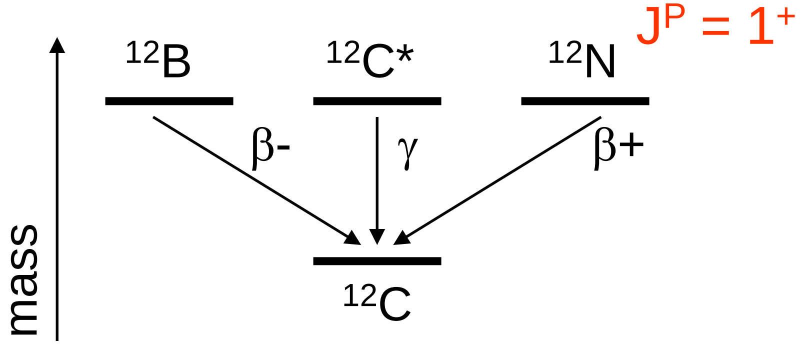
Weak interactions do not conserve I nor I_3 .

analogous to conservation of J and J_3 for angular momentum ⇒ selection rules

evidence that the strong force is invariant under isospin transformation:

physics unchanged by a symmetry operation

if in a system all protons are replaced by neutrons and all neutrons are replaced by protons!



nucleon – nucleon system

SU(2) algebra as for ordinary spin

(note that *pp* or *nn* systems have never been observed, the deuton composed of a proton and a neutron is therefore an ispospin singlet with $I = 0$)

$$|I=1, I_3=1\rangle = pp$$

$$|I=1, I_3=0\rangle = \sqrt{1/2}\left(pn+np\right)$$

$$|I=1, I_3=-1\rangle = nn$$

$$|I=0, I_3=0\rangle = \sqrt{1/2}\left(pn-np\right)$$

Not restricted to nucleons only

The 3 pion states π^+, π^0, π^- form an isospin triplet with $I = 1$.

Example

Compare the cross sections for the reactions

p + p → π^+ + d and

n + p → π^0 + d

(composition of angular momenta or isospin)

$$\sigma \sim |\,Amplitude\,|^2 \sim \sum_I |\langle I', I_3' | A | I, I_3 \rangle|^2$$

$$pp : |\tfrac{1}{2}, \tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle = |1,1\rangle \qquad \pi^+ d : |1,1\rangle |0,0\rangle = |1,1\rangle$$

$$np : |\tfrac{1}{2}, -\tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}}\,|1,0\rangle + \sqrt{\tfrac{1}{2}}\,|0,0\rangle \qquad \pi^0 d : |1,0\rangle |0,0\rangle = |1,0\rangle$$

isospin conservation in strong interactions

Therefore

$$\frac{\sigma\left(pp \to \pi^+ d\right)}{\sigma\left(np \to \pi^0 d\right)} = 2$$

(taken the other way around, as has been the case)

It can be used to show that the isospin on the pions is 1.

Symmetries and Conservation Laws

Suppose physics is invariant under the symmetry transformation U

$$\boxed{\psi \to \psi' = U\psi}$$

Conservation of probability requires

$$\langle \varphi | \psi \rangle = \langle \varphi' | \psi' \rangle = \langle U\varphi | U\psi \rangle = \langle \varphi | U^\dagger U | \psi \rangle$$

$\Rightarrow \boxed{U^\dagger U = 1}$ i.e. $U^{-1} = U^\dagger$ and $\det(U) = \pm 1$ i.e. U is unitary

For physical predictions to be unchanged by the symmetry transformation all matrix elements must remain unchanged

$$\boxed{\langle \varphi | H | \psi \rangle = \langle \varphi' | H | \psi' \rangle = \langle \varphi | U^\dagger H U | \psi \rangle}$$

i.e. require $\boxed{U^\dagger H U = H}$ $\Rightarrow$ $\boxed{[H, U] = 0}$

i.e. U commutes with the Hamiltonian.

Consider the infinitesimal transformation (ε small)

$\boxed{U = 1 + i\varepsilon G}$ G is called the generator of the transformation

From $U^{\dagger}U = (1 - i\varepsilon G)(1 + i\varepsilon G) = 1 + i\varepsilon(G - G^{\dagger}) + o(\varepsilon^2)$

neglecting terms in ε^2

unitarity implies $\boxed{G = G^{\dagger} = G^{-1}}$ G is hermitian, therefore an observable quantity

Furthermore $\left[H,U\right] = 0 \;\Rightarrow\; \left[H, 1 + i\varepsilon G\right] = 0 \;\Rightarrow\; \left[H,G\right] = 0$

i.e. the generator also commutes with the hamiltonian

and from QM $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle G\rangle = i\hbar\langle[H,G]\rangle = 0}$ (Ehrentest's theorem)

G is a conserved quantity

$\boxed{\text{symmetry} \quad \Leftrightarrow \quad \text{conservation law}}$

For each symmetry of nature there is an observable conserved quantity.

The finite transformation can be expressed as a series of infinitesimal transformations

$$U(\alpha) = \lim_{n\to\infty}\left(1 + i\frac{\alpha}{n}G\right)^{n} = e^{i\alpha G} \qquad \text{and} \qquad G = -i\left.\frac{\mathrm{d}U(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0}$$

In general the symmetry operation may depend on more than one parameter

$$\boxed{U = 1 + i\vec{\varepsilon}\cdot\vec{G}}$$

Isospin Symmetry of Strong Interactions

The strong interaction treats protons and neutrons (almost) equally → isospin symmetry, i.e. for the strong interaction nothing changes if all protons are replaced with neutrons and *vice versa*.

Isospin transformations described by SU(2) symmetry, in which the $\begin{pmatrix} p \\ n \end{pmatrix}$ doublet form the fundamental representation

$$\text{nucleon } |N\rangle = \begin{pmatrix} p \\ n \end{pmatrix} \quad \text{proton } |p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{neutron } |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the particle with the biggest charge (proton) has $I_3 = + ½$,

the particle with smallest charge (neutron) has $I_3 = - ½$

Express the invariance of the strong interaction under $p \leftrightarrow n$ transformation as invariance under “rotations” in an abstract isospin space

$$\begin{pmatrix} p' \\ n' \end{pmatrix} = U \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix}$$

The 2 × 2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters.

From unitarity $U^{\dagger}U = 1$ there are 4 constraints, and 8 – 4 = 4 independent matrices.

In the language of group theory the four matrices form the two dimensional unitary group *U(2)*.

Under this restriction, U has the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} e^{i\varphi} \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1$$

One of the matrices corresponds to multiplying by a phase factor $e^{i\varphi}$ ($\mathbf{I}e^{i\varphi}$), i.e.
a global phase transformation and not a flavour transformation, and is of no relevance.

The remaining three matrices form the special unitary group *SU(2)* with $\det(U) = +1$

and

$$\det(U) = +1 \;\; \Rightarrow \;\; Tr(G) = 0$$

A linearly independent choice for the generators G are the traceless Pauli spin matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Define the isospin generator $\boxed{\vec{I} = \tfrac{1}{2}\vec{\tau}}$ and the isospin transformation $\boxed{U = e^{i\vec{\alpha}\cdot\vec{I}}}$

The isospin generators satisfy the isospin algebra $\left[I_i, I_j\right] = i\varepsilon_{ijk} I_k$

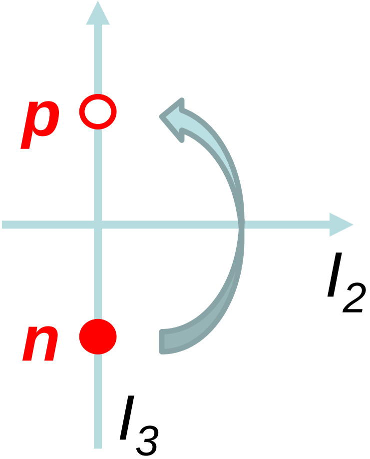
with $\quad I^2 = I_1^2 + I_2^2 + I_3^2 \quad$ and $\quad \left[I^2, I_i\right] = 0$

Nonlinear functions of the generators, which commute with all the generators,
are called Casimir operators (invariants) (*SU(2)* has rank 1 and 1 Casimir operator I^2).
Can diagonalize simultaneously I^2 and I_3 .

Can also define isospin ladder operators (useful for constructing higher order representations) similar to angular momentum

$$I_+ = I_1 + iI_2 \qquad I_+\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \qquad n \to p$$

$$I_- = I_1 - iI_2 \qquad I_-\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \qquad p \to n$$



$$I_+\left|I, I_3\right\rangle = \sqrt{I(I+1) - I_3(I_3+1)}\left|I, I_3+1\right\rangle$$

$$I_-\left|I, I_3\right\rangle = \sqrt{I(I+1) - I_3(I_3-1)}\left|I, I_3-1\right\rangle$$

i.e. increase or decrease the third component of the isospin by 1.

In summary:
the assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantities.

In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum.

Electromagnetic interactions conserve I_3 only (electric charge).

Weak interactions do not conserve I nor I_3.

Yang-Mills Gauge Theories

Invariance under isospin transformations in SU(2) space

$$\boxed{\psi \to \psi' = U\psi = e^{i\alpha_i \cdot \tau^i/2}\psi}$$

⇒ conservation of isospin *I* in strong interactions

Has isospin a dynamical role?
Can we transform a *p* into a *n* arbitrarily at any space-time point?
Can we build a theory of strong interactions from isospin invariance?
i.e. build a gauge invariant theory under local isospin transformations

$$\boxed{\psi \to \psi' = U\psi = e^{ig\alpha_i(x) \cdot \tau^i/2}\psi}$$

Following the EM template, introduce the covariant derivative

$$\boxed{\partial_\mu \to D_\mu = \partial_\mu + ig\frac{\tau_i}{2} \cdot B_\mu^i(x)}$$

with B^i_μ three new spin-1 vector fields, transforming as

$$\boxed{B_\mu^i(x) \to B_\mu^{i\,\prime}(x) = B_\mu^i(x) - \frac{1}{g}\partial_\mu \alpha^i(x) - i\;\varepsilon_{ijk}\;\alpha^j(x)\;B_\mu^k(x)}$$

This is the basic idea of Yang and Mills (1954), i.e. of non-abelian gauge theories.
Unfortunately, this theory is not supported by experiment, there are no such B^i_μ fields.
The idea used in the late ‘60s to develop the electroweak theory ($SU(2)_L \times U(1)_Y$)
and QCD ($SU(3)_c$ symmetry group) in 1973.

Discovery of the Pion (1947)

$\pi^+ \rightarrow \mu^+ \rightarrow e^+$ (cosmic rays)

points to note:

the pion decays at rest

dE/dx – Bragg Peak

the particle accompanying the μ^+ is not detected (ν_μ)

constant range for μ (~600μm) (i.e. 2-body decay)

low dE/dx for fast e^+ variable range

small angle scattering of tracks

first pion observed in emulsions

produced in interactions of cosmic rays in the upper atmosphere

Pion Spin

Consider the reaction $\pi^+ + d \to p + p$ and the inversed one $p + p \to \pi^+ + d$

and the cross sections

$$\frac{d\sigma}{d\Omega}\left(\pi^+ d \to pp\right) \propto \frac{1}{2}\frac{p_p}{p_\pi}\frac{1}{(2s_\pi+1)(2s_d+1)}\sum_{f,i}\left|M_{fi}\right|^2$$

$$\frac{d\sigma}{d\Omega}\left(pp \to \pi^+ d\right) \propto \frac{p_\pi}{p_p}\frac{1}{\left(2s_p+1\right)^2}\sum_{f,i}\left|M_{if}\right|^2$$

The detailed balance principle (time reversal invariance) requires

$$\sum_{f,i}\left|M_{fi}\right|^2 = \sum_{f,i}\left|M_{if}\right|^2$$

In the c.o.m. just above threshold (the thresholds energies are slightly different) that leads to

$$\frac{\sigma_{tot}\left(\pi^+ d \to pp\right)}{\sigma_{tot}\left(pp \to \pi^+ d\right)} \propto \frac{1}{2}\frac{\left(2s_p+1\right)^2}{(2s_\pi+1)(2s_d+1)}\left(\frac{p_p}{p_\pi}\right)^2 = \boxed{\frac{2}{3(2s_\pi+1)}\left(\frac{p_p}{p_\pi}\right)^2}$$

from measured values of this ratio, one obtains $2s_\pi + 1 = 0.97 \pm 0.31 \to s_\pi = 0$ (the deuton has spin 1 ħ)

Isospin: Consider the reactions $p + p \to \pi^+ + d$ and $n + p \to \pi^0 + d$ (slide 6).

Isospin Representation of the Pion

Pions form an ispospin triplet with *I = 1* : $\pi^{+}\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\pi^{0}\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\pi^{-}\begin{pmatrix}0\\0\\1\end{pmatrix}$

The third component of isospin

$$I_3\left|\pi^{+}\right\rangle = +1\left|\pi^{+}\right\rangle$$
$$I_3\left|\pi^{0}\right\rangle = 0\left|\pi^{0}\right\rangle \qquad \Rightarrow \boxed{I_3 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}}$$
$$I_3\left|\pi^{-}\right\rangle = -1\left|\pi^{-}\right\rangle$$

To find I_1, I_2 first consider the ladder operators $I_+ = I_1 + iI_2$ and $I_- = I_1 - iI_2$:

$$I_+\left|\pi^{-}\right\rangle = I_+\begin{pmatrix}0\\0\\1\end{pmatrix} = \sqrt{2}\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad I_+\left|\pi^{0}\right\rangle = I_+\begin{pmatrix}0\\1\\0\end{pmatrix} = \sqrt{2}\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad I_+\left|\pi^{+}\right\rangle = I_+\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

from which follows the matrix representation for I_+ and I_-

$$I_+ = \begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix} \qquad I_- = \begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}$$

and solve for I_1 and I_2

$$\boxed{I_1 = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad I_2 = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}}$$

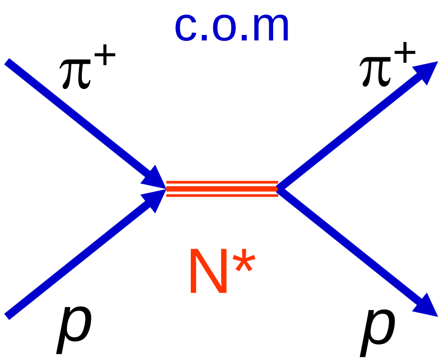
Production of Resonances

~1950 first accelerator beams
Can produce π^+/π^- beams

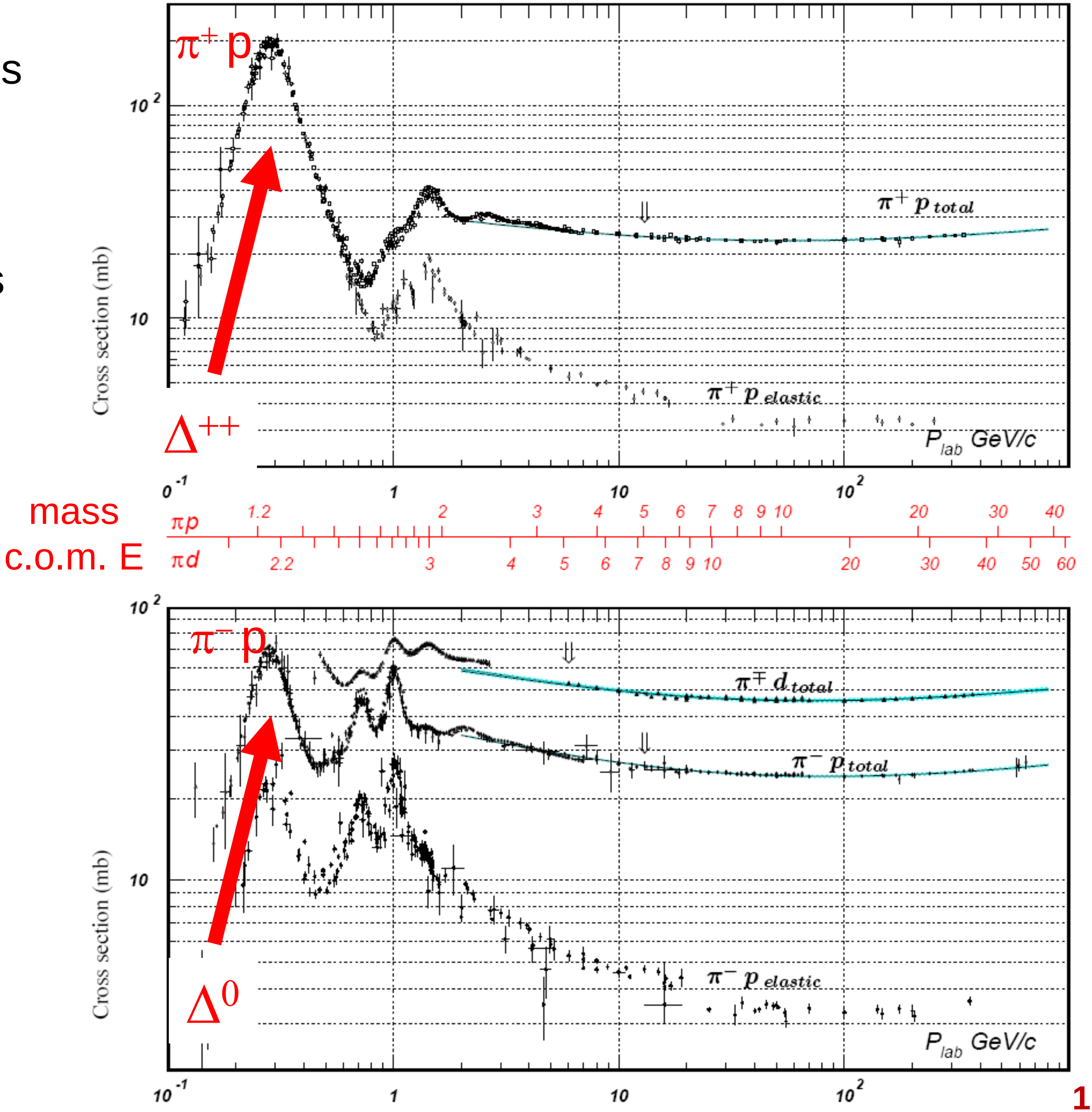
In 1951 Fermi discovers the
spin 3/2 Δ(1232) resonances

Δ^{++} Δ^{+} Δ^{0} Δ^{-}

in πp scattering:



Afterwards,
many more resonances
have been discovered.



Example

There are four Δ states with *I = 3/2*, I_3 (Δ^{++}) = +3/2, …, I_3 (Δ^{-}) = -3/2

Experimentally $$\frac{\sigma_{tot}\left(\pi^+ p\right)}{\sigma_{tot}\left(\pi^- p\right)} = 3$$ at the resonance peak.

Taking into account the isospin of the πp system and that of the Δ

$$\pi^+ p : |1,1\rangle |{}^1\!/_2,{}^1\!/_2\rangle = |{}^3\!/_2,{}^3\!/_2\rangle \qquad \Delta^{++} : |{}^3\!/_2,+{}^3\!/_2\rangle$$

$$\pi^- p : |1,-1\rangle |{}^1\!/_2,{}^1\!/_2\rangle = \boxed{\sqrt{\frac{1}{3}}}\,|{}^3\!/_2,-{}^1\!/_2\rangle + \sqrt{\frac{2}{3}}\,|{}^1\!/_2,-{}^1\!/_2\rangle \qquad \Delta^{0} : |{}^3\!/_2,-{}^1\!/_2\rangle$$

isospin conservation

we can explain this ratio.

How can we determine the spin of the Δ ?
From the angular distribution of the *p* – π system (*p*-wave)

Resonances and Particles

Particles that decay by the strong interaction are extremely short lived (~10^{-23} s). They are called resonances and are identified by observing their decay products or "bumps" in the cross section, as a function of the energy of the system, i.e. $\sigma(E)$

$$\pi p \rightarrow \Delta \rightarrow \pi p$$

The resonances are too short lived to determine precisely their mass (energy), which is "spread" around a central value E_R .

According to Heisenberg $\Delta E \cdot \Delta t \geq \frac{\hbar}{2} \quad \Rightarrow \quad \Gamma \tau = \hbar$

Γ is the width of the resonance

This unstable particle decays according to the exponential law

$$N = N_o e^{-t/\tau}$$

with $\tau = \hbar / \Gamma$ the "lifetime" of the sate.

The time evolution of the particle's wave function must include the "decay factor" Γ and can be rewritten with the substitution $\boxed{m \rightarrow m - i\Gamma/2}$

$$\psi(t) = \psi(0) e^{-imt} \rightarrow \psi(0) e^{-imt} e^{-\Gamma t/2}$$

$$\Longrightarrow \boxed{|\psi(t)|^2 = |\psi(0)|^2 e^{-\Gamma t}}$$

Breit-Wigner Resonance

The state can be described by the Fourier transform of the time dependent wave function

$$\chi(E)=\int_{-\infty}^{+\infty}\psi(t)e^{iEt}dt=\int_{0}^{+\infty}\psi(0)e^{-t[(\Gamma/2)+i(E_R-E)]}dt=\frac{K}{(E-E_R)+i\,\Gamma/2}$$

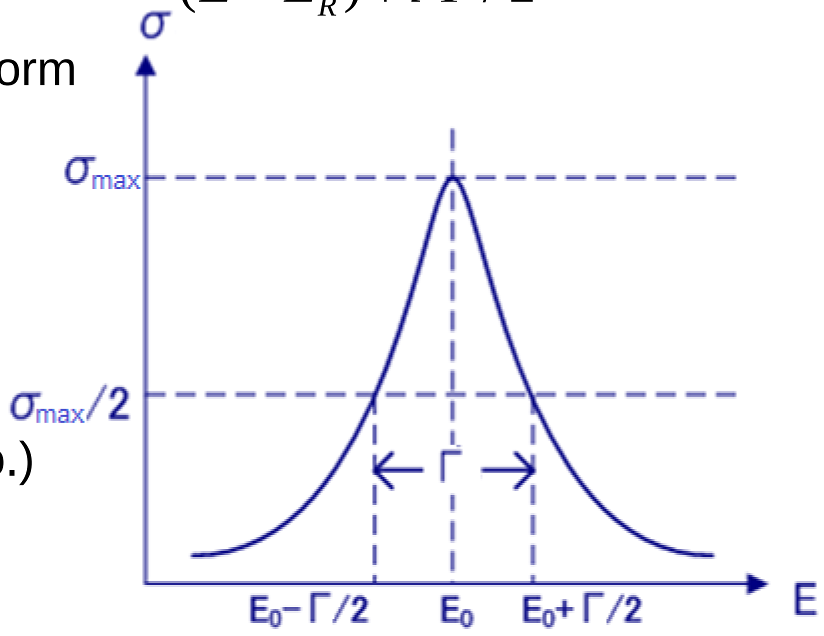
One then observes a πp reaction rate $\sigma(E)$ of the form ($\sigma(E) \propto \chi^*\chi$)

$$\boxed{\sigma(E)=\sigma_{\max}\frac{\Gamma^2/4}{(E-E_R)^2+\Gamma^2/4}}$$

known as the Breit – Wigner resonance

with $\sigma_{\max}=4\pi\lambda\!\!\!^{-2}=4\pi\frac{\hbar^2}{p^2}$ (s – wave) (opt. theo.)

The total width Γ of the resonance is related to the strength of the interaction.



The resonance can decay to a number of different final states. Each individual decay mode has a partial width Γ_i .

Sum of all “partial widths” = total width $\Gamma=\sum_i \Gamma_i$

The shape is the same for all decays.

Branching Ratio: fraction of decays to the final state *i* BR = Γ_i / Γ

To obtain a relativistically invariant expression, multiply denominator and numerator by *(E + E*$_R$*)*^{2}

$$\sigma(E)=\sigma_{\max}\frac{\Gamma^2/4}{(E-E_R)^2+\Gamma^2/4}\frac{(E+E_R)^2}{(E+E_R)^2}$$

and by noting that around the peak $E \sim E_R$

$$\boxed{\sigma(E)=\sigma_{\max}\frac{\Gamma^2E_R^2}{\left(s-E_R^2\right)^2+\Gamma^2E_R^2}}$$

The particle can decay to a number of different final states (also different initial states). The shape, not the width, is the same for all decay modes:

$$T_{fi}=\sum\nolimits_n\frac{\langle f|H_I|n\rangle\langle n|H_I|i\rangle}{E-E_n}\sim\frac{\langle f|H_I|R\rangle\langle R|H_I|i\rangle}{E-\left(E_R-i\Gamma/2\right)}\propto\frac{\sqrt{\Gamma_f}\sqrt{\Gamma_i}}{E-\left(E_R-i\Gamma/2\right)}$$

Moreover, we have to take into account also the spin multiplicity factors (sum over final spins, average over initial spin states

$$\boxed{\begin{aligned}&\sigma_{if}=\frac{4\pi\hbar^2}{p_i^2}\left(\frac{(2J+1)}{(2s_1+1)(2s_2+1)}\right)\left(\frac{\Gamma_i\Gamma_fM_R^2}{(s-M_R^2)^2+\Gamma^2M_R^2}\right)\\&p_i=\text{intial momentum}\\&J=\text{resonance spin},\quad s_1,s_2=\text{spins of incoming particles}\\&\Gamma_\text{i},\Gamma_f=\text{widths of initial and final states, }\Gamma=\text{total width}\end{aligned}}$$

Cross-section Upper Bound and σ_{MAX}

Let $\psi_{in} = Ae^{ikz}$ be the incoming wave

and the scattered wave $\psi_{out} \approx A\left\{e^{ikz} + f(\vartheta)\frac{e^{i\vec{k}\cdot\vec{r}}}{r}\right\}$ for large *r*

$f(\vartheta)$ is the scattering amplitude and $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\vartheta)|^2$

Expand $f(\vartheta)$ with Legendre polynomials $f(\vartheta) = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\vartheta)$

where $a_l(k)$ is the partial wave amplitude $a_l(k) = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{1}{k}e^{i\delta_l}\sin\delta_l$

δ_l - phase shift

Then $\sigma = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l(k)|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$

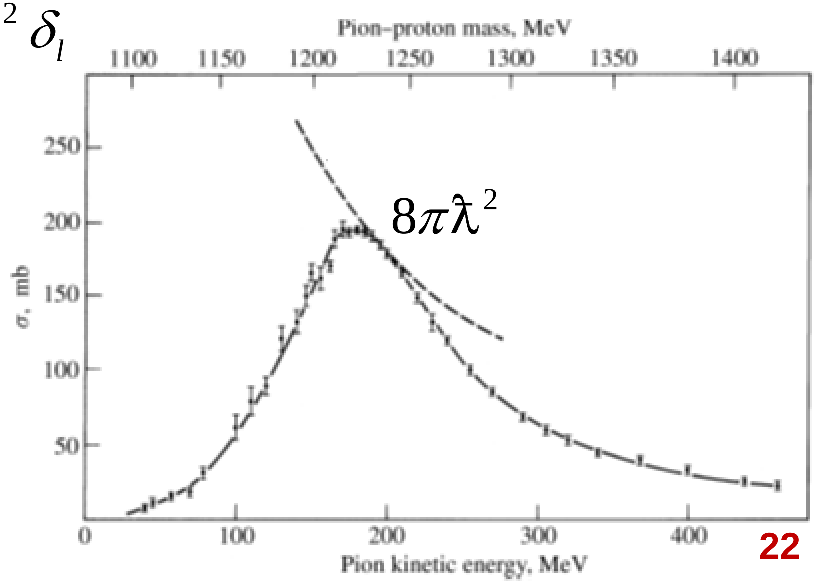
Since $\sin\delta_l \le 1 \Rightarrow \sigma_{MAX}^{(l)} = \frac{4\pi}{k^2}(2l+1)$
for partial wave *l*

$k^2 = E_{CM}^2$

At high energy
the state is in an s – wave

$$\sigma_{MAX} = \frac{4\pi}{k^2}$$

and include initial / final state spin multiplicity



Isospin of Anti-Nucleons

Anti-nucleon states are obtained by applying the charge conjugation operator *C* to the (p,n) doublet.

Consider the isopsin transformation $\begin{pmatrix} p' \\ n' \end{pmatrix} = U \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \alpha\, p + \beta\, n \\ -\beta^* p + \alpha^* n \end{pmatrix}$

and the charged conjugated transformation

$$\begin{pmatrix} \overline{p}' \\ \overline{n}' \end{pmatrix} = C \begin{pmatrix} p' \\ n' \end{pmatrix} = CU \begin{pmatrix} p \\ n \end{pmatrix} = U^* C \begin{pmatrix} p \\ n \end{pmatrix} = U^* \begin{pmatrix} \overline{p} \\ \overline{n} \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \overline{p} \\ \overline{n} \end{pmatrix} = \begin{pmatrix} \alpha^* \overline{p} + \beta^* \overline{n} \\ -\beta\, \overline{p} + \alpha\, \overline{n} \end{pmatrix}$$

We would like that the anti-nucleon doublet to transform in exactly the same way as the nucleon doublet to combine particle and anti-particle states in the same way,

i.e. $\boxed{\overline{\psi}' = U \overline{\psi}}$

Rearranging

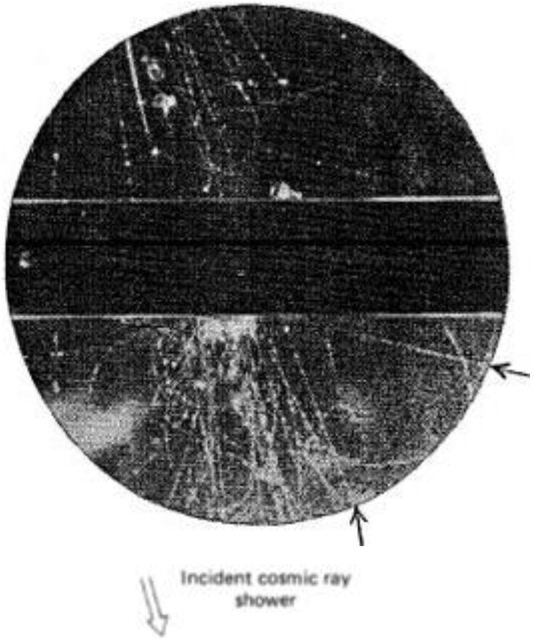
$$\begin{cases} \overline{p}' = \alpha^* \overline{p} + \beta^* \overline{n} \\ \overline{n}' = -\beta\, \overline{p} + \alpha\, \overline{n} \end{cases} \Rightarrow \begin{cases} -\overline{n}' = \alpha(-\overline{n}) + \beta\, \overline{p} \\ \overline{p}' = -\beta^*(-\overline{n}) + \alpha^* \overline{p} \end{cases} \Rightarrow \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} -\overline{n} \\ \overline{p} \end{pmatrix} = U \begin{pmatrix} -\overline{n} \\ \overline{p} \end{pmatrix}$$

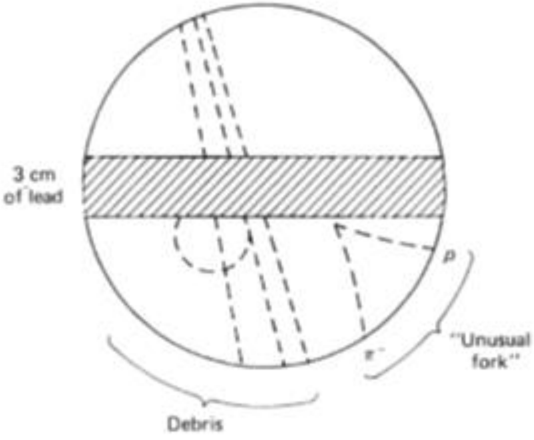
which means that the isodoublet $\begin{pmatrix} -\overline{n} \\ \overline{p} \end{pmatrix}$ transforms in the same way as $\begin{pmatrix} p \\ n \end{pmatrix}$

1. the anti-particle with biggest charge – the antineutron – has $I_3 = + ½$
2. introduce a minus sign for the upper component, the antineutron

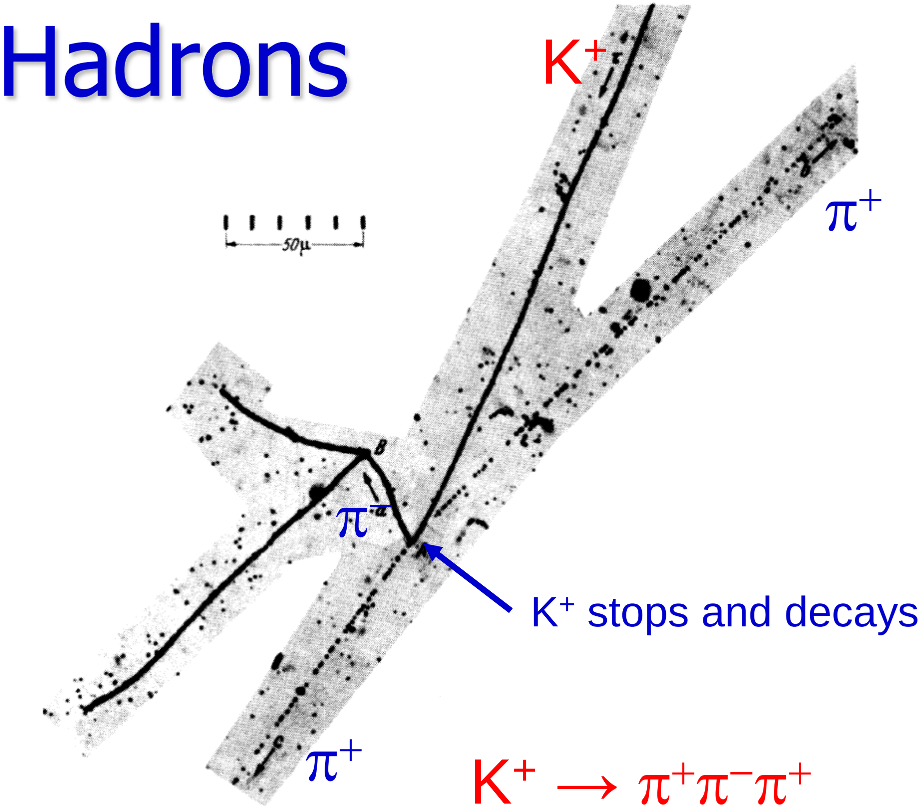
Note that this works only with *SU(2)*.

Discovery of Strange Hadrons

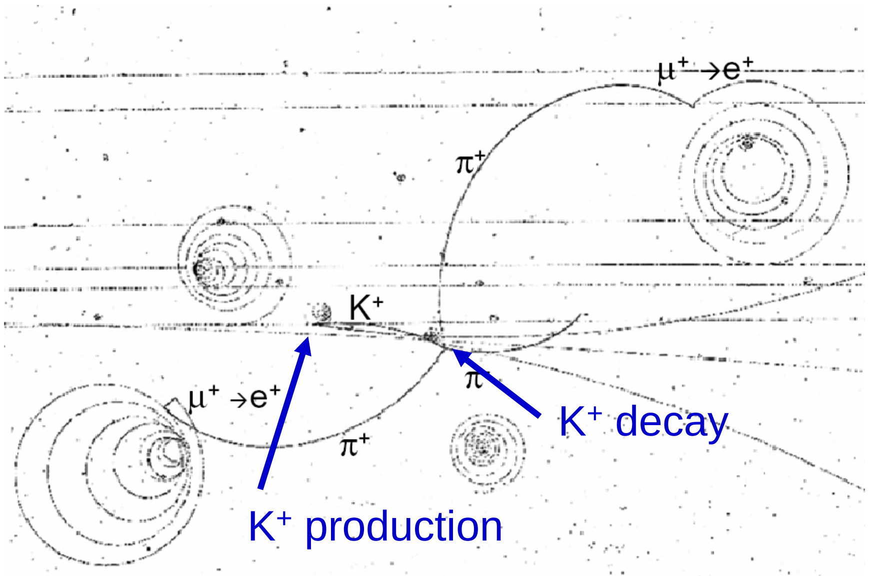




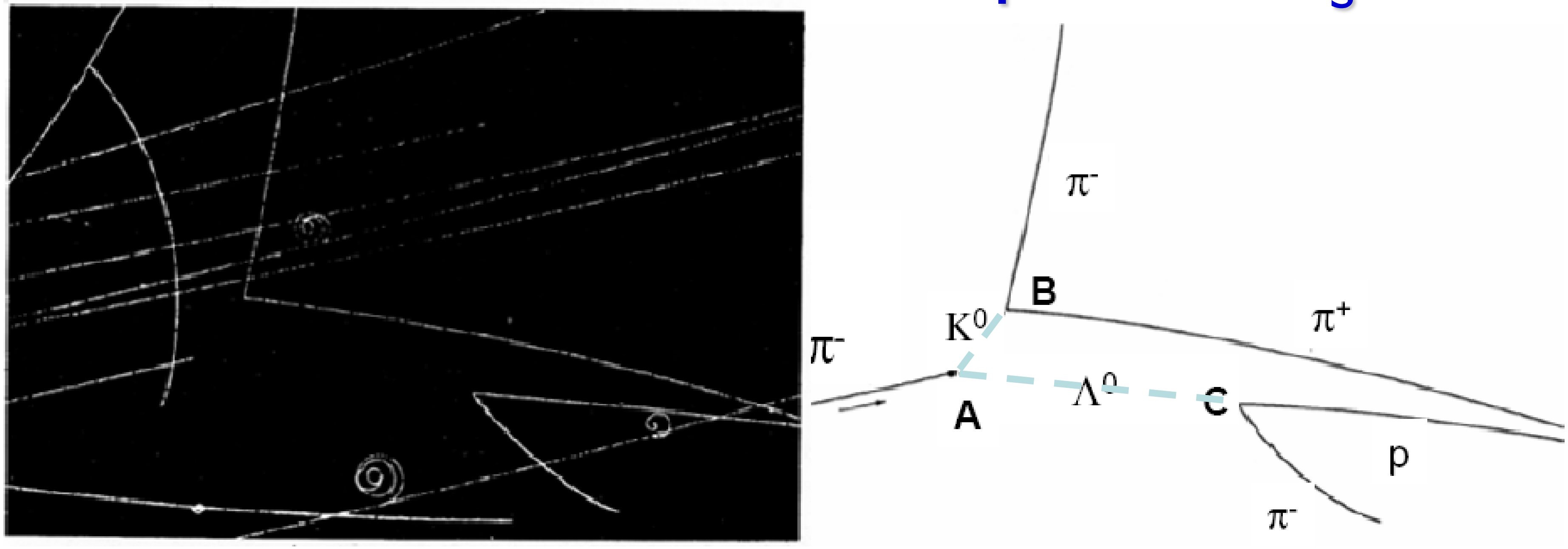
$\Lambda \rightarrow p\pi^-$



$K^+ \rightarrow \pi^+\pi^-\pi^+$



Associated Production : $\pi^- p \to \Lambda K^0_s$



strange particles are always produced in pairs (associated production):
strange and anti-strange hadron (i.e. strange quark pairs s and $\bar{s}$)

Gell-Mann and Nishijima (1963): evidence of a new quantum number – strangeness *S*

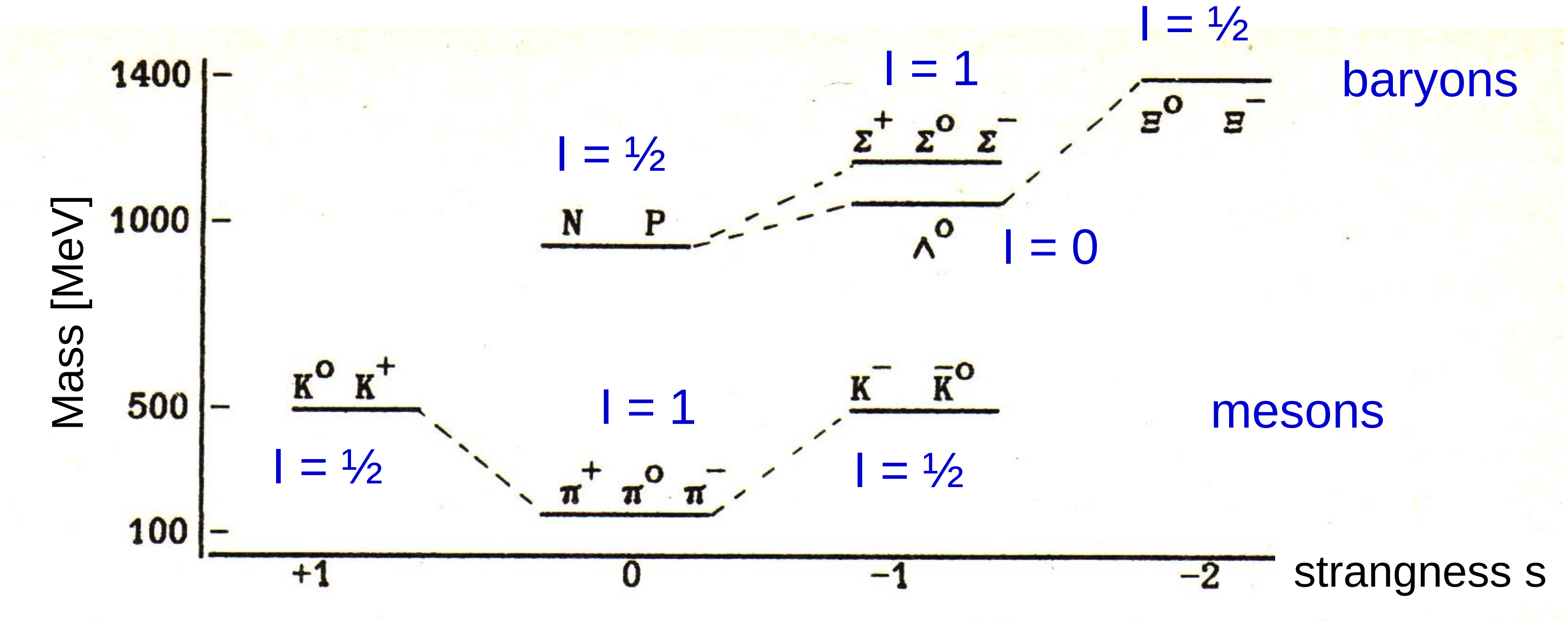
S = 0 π, N, Δ, …

S = 1 K^+, …

S = -1 Λ, Σ, …

strangness is conserved in strong and electromagnetic interactions
strangness is violated in weak (hadronic or semileptonic) decays
 $s \to u + \ldots$ $\Delta S = \Delta Q$

Strange Particles (circa ’55)



Some regularities are visible :
hadrons can be grouped in multiplets of similar mass (isospin multiplets)
and same quantum numbers: spin, parity, isospin

Hint of some underlying symmetry →
extension of isospin to include also strange hadrons
The symmetry, however, is only approximate: m_p = 938 MeV to m_Ξ = 1320 MeV.

Introduction of Quarks

1969

Hadrons are extended objects and have structure:

anomalous magnetic moments (~´30s), i.e. not perfect Dirac particles

proliferation of hadrons (~´50s)

regularities in hadron spectrum

elastic electron-nucleon scattering (~´50s)

⇒ Hadrons are not elementary particles but composed of quarks u, d, s (Gell-Mann 1964)

baryons are composed of three quarks (B = 1)

mesons are composed of quark – anti-quark pairs (B = 0)

anti-baryons are composed of three anti-quarks (B = -1)

Quarks carry fractional electric charges

$Q_u = 2/3$ e

$Q_d = -1/3$ e

$Q_s = -1/3$ e

have spin 1/2 ħ and baryon number 1/3, $m_u \approx m_d \sim 340$ MeV, $m_s \sim 500$ MeV

By adding up quark’s quantum #s one obtains the hadron’s quantum #s.

Since $m_s > m_u \approx m_d$, we do not have an exact symmetry, m_s not so different from m_u, m_d

⇒ can treat the hadron states as if they were symmetric under $u \leftrightarrow d \leftrightarrow s \leftrightarrow u$

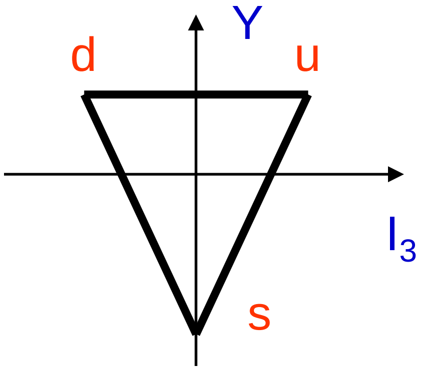
⇒ assume charge independence of the strong force

Any results obtained from this assumption are only approximate (symmetry not exact).

With the introduction of a second additive quantum number *S* (strangeness), enlarge SU(2) isospin symmetry to a larger group of rank 2 → $SU(3)_F$.
$SU(3)_F$ flavor symmetry is far from being an exact symmetry, but allows us to organize known hadrons into multiplets with same quantum numbers J^P, and isospin sub-multiplets
This allows us to classify all observed hadrons,
to predict some of their properties, and to predict new hadron states.

Introduce a new additive quantum number, the hypercharge Y to make the hadron multiplets look more symmetric (nothing deeper behind it)

$$Y = B + S \quad \rightarrow \quad Q = I_3 + \tfrac{1}{2}Y$$



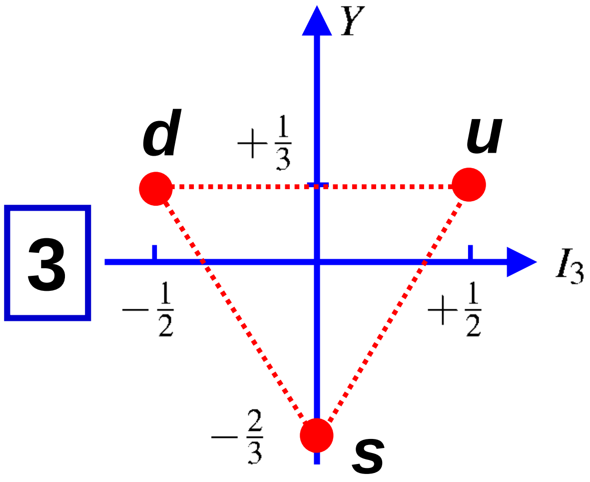
The (u,d,s) multiplets represents the fundamental representation of the $SU(3)_F$ flavor group from which all other multiplets can be built.
In group theory language

baryons: $3 \otimes 3 \otimes 3$

mesons: $3 \otimes \bar{3}$

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Quarks and anti-Quarks in *SU(3)$_F$* Flavor

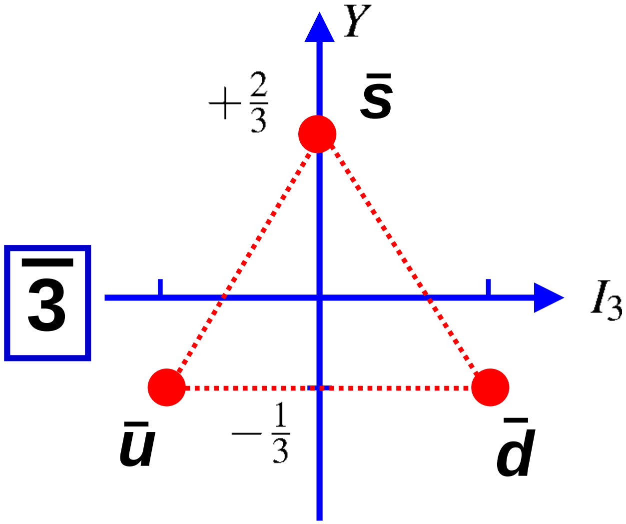


Quarks

$$I_3 u = +\tfrac{1}{2}u; \quad I_3 d = -\tfrac{1}{2}d; \quad I_3 s = 0$$

$$Y u = +\tfrac{1}{3}u; \quad Y d = +\tfrac{1}{3}d; \quad Y s = -\tfrac{2}{3}s$$

The anti-quarks have opposite *SU(3)$_F$* flavour quantum numbers



Anti-Quarks

$$I_3 \overline{u} = -\tfrac{1}{2}\overline{u}; \quad I_3 \overline{d} = +\tfrac{1}{2}\overline{d}; \quad I_3 \overline{s} = 0$$

$$Y \overline{u} = -\tfrac{1}{3}\overline{u}; \quad Y \overline{d} = -\tfrac{1}{3}\overline{d}; \quad Y \overline{s} = +\tfrac{2}{3}\overline{s}$$

Additive Quantum Numbers of Quarks

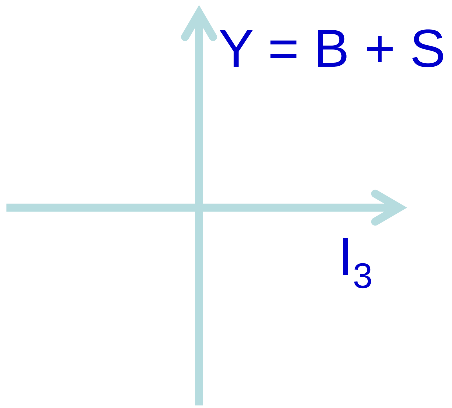
Quark Property	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	-1	0	0	0
C – charm	0	0	0	$+1$	0	0
B – bottomness	0	0	0	0	-1	0
T – topness	0	0	0	0	0	$+1$

constituent quark masses in quark model:

m_u ~ 336 MeV, m_d ~ 340 MeV, m_s ~ 485 MeV
m_c ~ 1,550 MeV, m_b ~ 4,730 MeV, m_t ~ 177,000 MeV

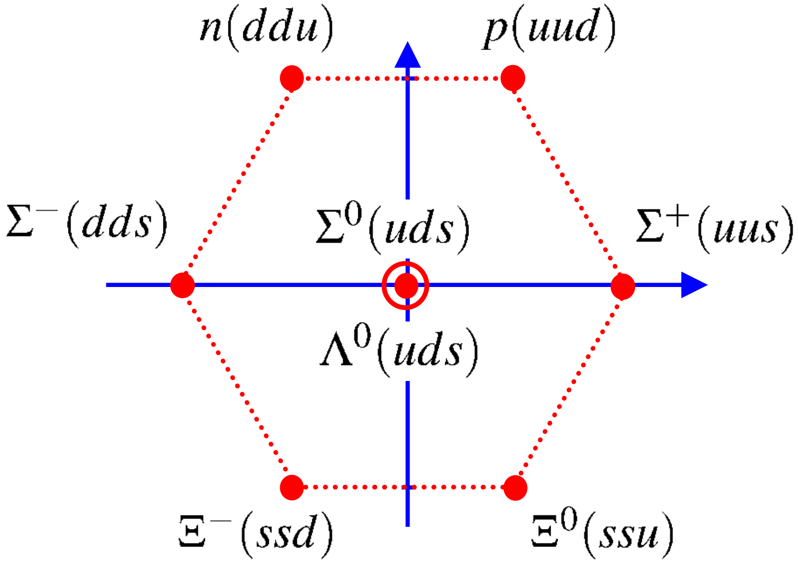
Baryon Octet and Meson Nonet

The eight-fold way



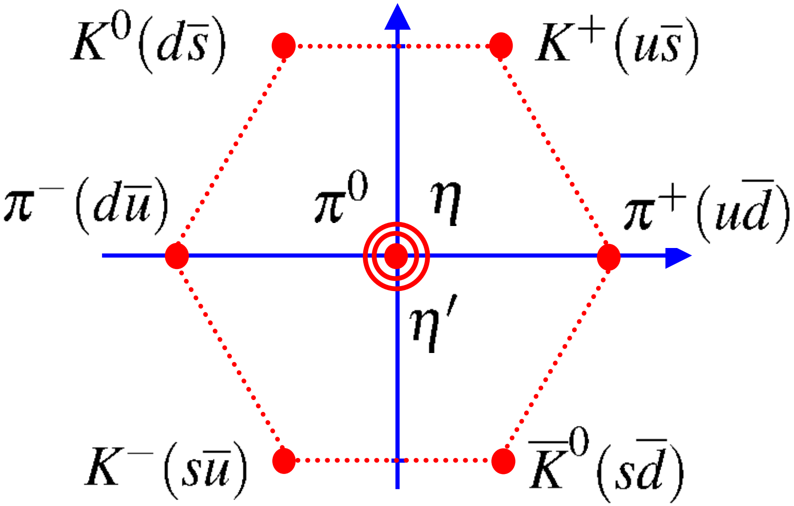
BARYON OCTET

(L=0, S=1/2, J=1/2, P= +1)



PSEUDOSCALAR MESON NONET

(L=0, S=0, J=0, P= –1)



SU(3)$_F$ Flavor

The postulated u d s flavour symmetry can be expressed as (recall isospin)

$$\boxed{\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}}$$

The 3 × 3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters.

There are 9 constraints from unitarity $\hat{U}^\dagger \hat{U} = 1$

Can form 18 – 9 = 9 linearly independent matrices. These 9 matrices form a *U(3)* group.
One matrix is the identity multiplied by a complex phase (***I***$e^{i\varphi}$) and is of no interest.

The remaining 8 matrices have det *U* = +1 and form an *SU(3)* group.

Introduce the 8 generators T_a of the SU(3) by considering the infinitesimal transformation

$$\hat{U} = 1 + i\sum_a \varepsilon_a T_a \qquad (\vec{\varepsilon} = \vec{\alpha} / n,\ n \rightarrow \infty)$$

A generic element of the group can be written as $\hat{U} = e^{i\vec{\alpha}\cdot\vec{T}} = e^{i\sum_a \alpha_a T_a}$

with $T_a = \frac{1}{2}\lambda_a$ the eight hermitian generators of the *SU(3)* group

and the 8 λ_a Gell-Mann matrices (equivalent to the Pauli spin matrices for *SU(2)*)
α_a are 8 “rotation angles” in the *SU(3)* space.

The Gell-Mann Matrices

In $SU(3)_F$ flavour, the three quark states are represented by:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The *SU(3)* u d s flavour symmetry "contains" u ↔ d, u ↔ s, and d ↔ s SU(2) symmetrie

The *SU(2)* u ↔ d flavour symmetry allows us to represent the first three matrices as:

$$\lambda_1 = \begin{pmatrix} \multicolumn{2}{c}{\tau_1} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} \multicolumn{2}{c}{\tau_2} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} \multicolumn{2}{c}{\tau_3} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. **u ↔ d**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The third component of isospin is now written $I_3 = \frac{1}{2}\lambda_3$

with $\quad I_3\ u = +\frac{1}{2}u \qquad I_3\ d = -\frac{1}{2}d \qquad I_3\ s = 0$

I_3 "counts the number of up quarks minus the number of down quarks in a state.

Similarly, the matrices corresponding to the SU(2) u ↔ s and d ↔ s symmetries can be represented as

u ↔ s $$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

d ↔ s $$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

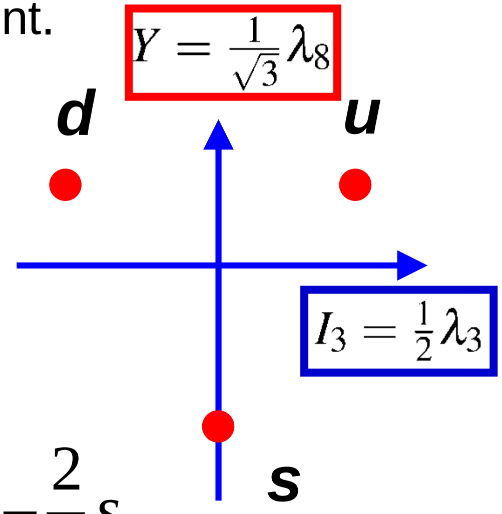
In addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we have two other traceless diagonal matrices λ_X and λ_Y.

However the three diagonal matrices are not linearly independent.
Define the eighth matrix, λ_8 , as the linear combination

$$\lambda_8 = \frac{1}{\sqrt{3}}\lambda_X + \frac{1}{\sqrt{3}}\lambda_Y = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which determines the hypercharge $Y = \frac{1}{\sqrt{3}}\lambda_8$

$$Y\ u = +\frac{1}{3}u \qquad Y\ d = +\frac{1}{3}d \qquad Y\ s = -\frac{2}{3}s$$



SU(3) Ladder Operators

Consider the u ↔ s symmetry “V-spin” to which we can associate the u → s and u → s ladder operators

$$V_{\pm} = \frac{1}{2}\left(\lambda_4 \pm i\lambda_5\right) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \pm \frac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

with

$$V_{+}s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u \quad V_{+}u = 0 \quad V_{+}d = 0$$

$$V_{-}u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = +s \quad V_{-}d = 0 \quad V_{-}s = 0$$

The actions of the six ladder operators are:

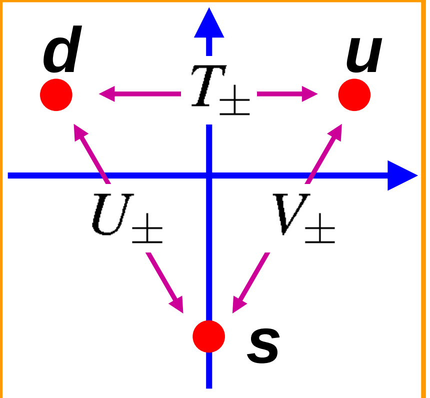
$T_{+}d = u; \quad T_{-}u = d;$	$T_{+}\bar{u} = -\bar{d}; \quad T_{-}\bar{d} = -\bar{u}$
$V_{+}s = u; \quad V_{-}u = s;$	$V_{+}\bar{u} = -\bar{s}; \quad V_{-}\bar{s} = -\bar{u}$
$U_{+}s = d; \quad U_{-}d = s;$	$U_{+}\bar{d} = -\bar{s}; \quad U_{-}\bar{s} = -\bar{d}$

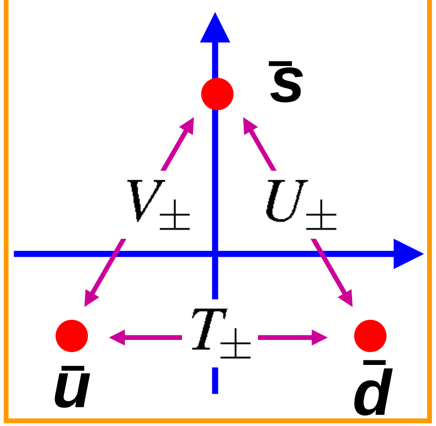
SU(3) LADDER OPERATORS

$$T_{\pm} = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$

$$V_{\pm} = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$

$$U_{\pm} = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$





Gell-Mann Matrices

u ↔ d

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

u ↔ s

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

d ↔ s

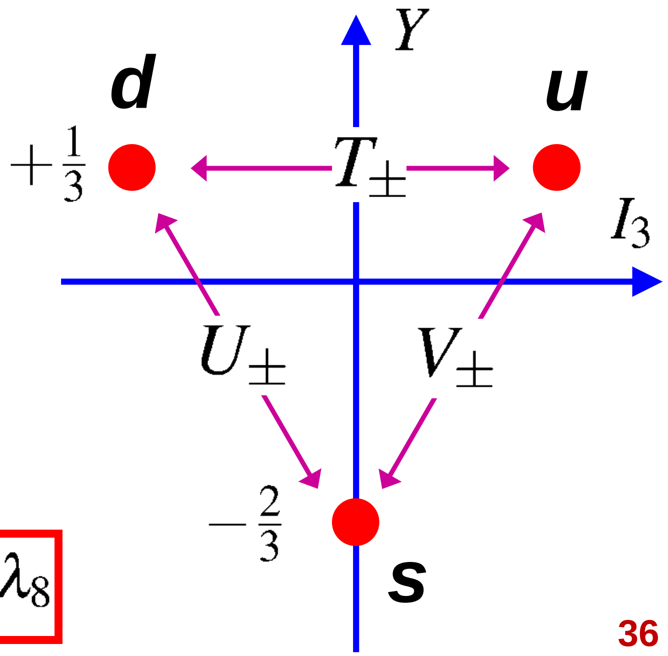
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and the ladder operators which step up / down between the states

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$

with isospin and hypercharge $\boxed{I_3 = \tfrac{1}{2}\lambda_3}$ $\boxed{Y = \frac{1}{\sqrt{3}}\lambda_8}$



SU(3) Algebra

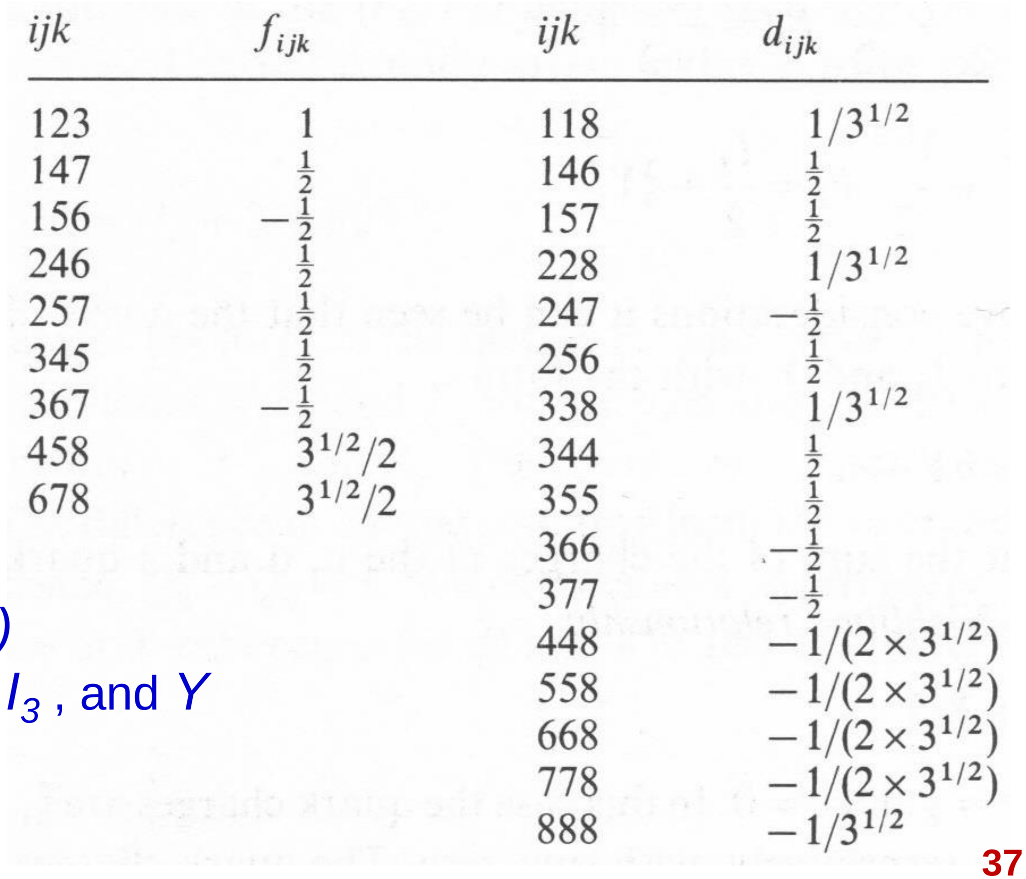
The *SU(3)* generators are the 8 λ_a traceless Gell-Mann matrices which do not commute

$$\left[\lambda_a,\lambda_b\right]=2if_{abc}\lambda_c \qquad Tr\left(\lambda_a\lambda_b\right)=2\delta_{ab} \qquad Tr\left(\lambda_a\right)=0$$

$$\left\{\lambda_a,\lambda_b\right\}=\frac{4}{3}\delta_{ab}\lambda_3+2\sum_c d_{abc}\lambda_c$$

f_{ijk} – anti-symmetric structure constant of *SU(3)* group

d_{ijk} – symmetric structure constant of *SU(3)* group



ijk	f_{ijk}	ijk	d_{ijk}
123	1	118	$1/3^{1/2}$
147	$\frac{1}{2}$	146	$\frac{1}{2}$
156	$-\frac{1}{2}$	157	$\frac{1}{2}$
246	$\frac{1}{2}$	228	$1/3^{1/2}$
257	$\frac{1}{2}$	247	$-\frac{1}{2}$
345	$\frac{1}{2}$	256	$\frac{1}{2}$
367	$-\frac{1}{2}$	338	$1/3^{1/2}$
458	$3^{1/2}/2$	344	$\frac{1}{2}$
678	$3^{1/2}/2$	355	$\frac{1}{2}$
		366	$-\frac{1}{2}$
		377	$-\frac{1}{2}$
		448	$-1/(2\times3^{1/2})$
		558	$-1/(2\times3^{1/2})$
		668	$-1/(2\times3^{1/2})$
		778	$-1/(2\times3^{1/2})$
		888	$-1/3^{1/2}$

rank of *SU(3)* is two

→ two Casimir operators: I^2 and Y *(S)*

→ can diagonalize simultaneously I^2 , I_3 , and Y

Mesons: $q\bar{q}$ States

Mesons are composed of a quark and an antiquark bound together.
The mesons' quantum numbers are obtained by adding up those of the $q\bar{q}$ pair.
Meson states can be obtained by combining two fundamental representations of the *SU(3)$_F$* group

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} = 3 \otimes \bar{3} = 8 \oplus 1$$

therefore there are nine states – mesons grouped in an octet and a singlet under *SU(3)$_F$*.
Let us start with two flavors, u and d (4 states), and add later the quark s;

we obtain an isotriplet I = 1

$$|I=1, I_3=1\rangle = -u\bar{d} = \pi^+$$

$$|I=1, I_3=0\rangle = \sqrt{1/2}\left(u\bar{u} - d\bar{d}\right) = \pi^0$$

$$|I=1, I_3=-1\rangle = d\bar{u} = \pi^-$$

and isosinglet I = 0

$$|I=0, I_3=0\rangle = \sqrt{1/2}\left(u\bar{u} + d\bar{d}\right) \approx \eta^0$$

Now let's us add the strange quark.
Six states are combinations of a quark and an antiquark of different flavor:

$u\bar{d}$, $d\bar{s}$, $s\bar{u}$, $u\bar{s}$, $d\bar{u}$, $s\bar{d}$

Three sates are formed of combinations of quark and antiquarks of same flavor

and have $I_3 = Y = 0$: $u\bar{u}$, $d\bar{d}$, $s\bar{s}$

If the $SU(3)_F$ flavour symmetry were exact, the choice of states wouldn't matter.
One of the states has an equal admixture of $u\bar{u}$, dd and $s\bar{s}$ quarks.

$$\boxed{\sqrt{1/3}\left(u\bar{u}+d\bar{d}+s\bar{s}\right)=\eta_1\approx\eta'}$$

It is flavorless in the sense that is a singlet under $SU(3)_F$ flavor transformations: $U\eta_1=\eta_1$:

$$T_+\eta_1=T_-\eta_1=U_+\eta_1=U_-\eta_1=V_+\eta_1=V_-\eta_1=0$$

Experimentally observe three light mesons with m ~ 140 MeV: π^+, π^0, π^-
Identify one state (the π^0) with the isospin triplet

$$\boxed{\sqrt{1/2}\left(u\bar{u}-d\bar{d}\right)=\pi^0}$$

The third state can be obtained by taking the linear combination of the other two $q\bar{q}$ states which is orthogonal to the π^0 and to the η_1

$$\boxed{\sqrt{1/6}\left(u\bar{u}+d\bar{d}-2s\bar{s}\right)=\eta_8\approx\eta^0}$$

Because $SU(3)_F$ flavour is only approximate the physical states with $I_3 = 0$, $Y = 0$ can be mixtures of the octet and singlet states
(if SU(3) symmetry were exact, the choice of the states would not matter):

$$\eta^0=\eta_8\sin\vartheta+\eta_1\cos\vartheta$$

$$\eta'=\eta_8\cos\vartheta-\eta_1\sin\vartheta$$

The mixing has to be determined experimentally:
$\theta\approx$ -25^0 for pseudoscalar and $\theta\approx$ 35^0 for vector mesons

Meson Quantum Numbers

Mesons' quantum numbers: multiplets are classified according to *J*, *P*, and *C* : J^{PC}

SPIN (or total angular momentum): J = S + L

Mesons composed of 2 spin ½ quarks, S = 0 or S = 1, with orbital angular momentum L

PARITY

Parity P (space inversion $\theta \to \pi - \theta$, $\phi \to \phi + \pi$, or reflection through origin)

$$\boxed{P = P_q \cdot P_{\bar{q}} \cdot (-1)^L = -1 \cdot (-1)^L = (-1)^{L+1}}$$

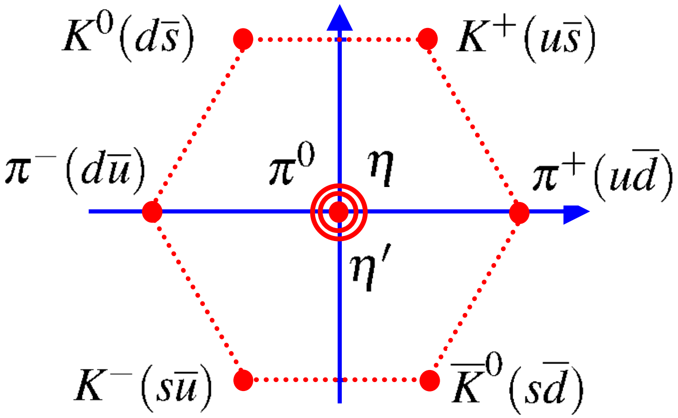
(q and $\bar{q}$ have opposite intrinsic parity)

CHARGE CONJUGATION

Charge conjugation C (a neutral state can be an eigenstate of C, i.e. a $q\bar{q}$ state like a π^0)

$$\boxed{C = -1 \cdot (-1)^{S+1} \cdot (-1)^L = (-1)^{L+S}}$$

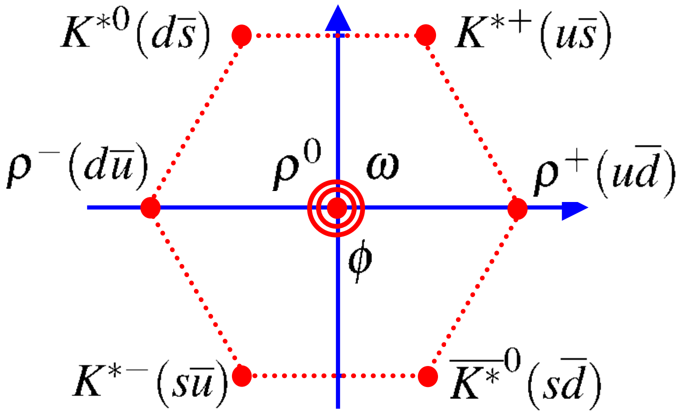
PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1, C= +1, $J^{PC} = 0^{-+}$)



In the pseudoscalar mesons the spins are anti- aligned.

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

PSEUDOVECTOR MESONS (L=0, S=1, J=1, P= –1, C= –1, $J^{PC} = 1^{--}$))



In the pseudovector mesons the spins are aligned. The physical states are found to be approximately "ideally mixed":

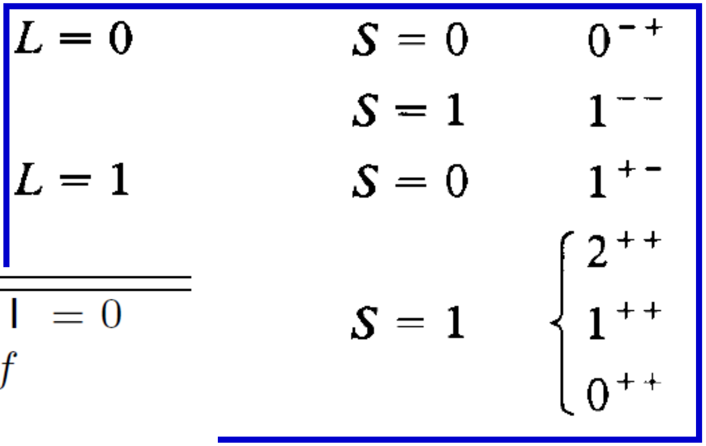
$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

MASSES

$\pi^\pm : 140\,\text{MeV}$ $\quad\pi^0 : 135\,\text{MeV}$
$K^\pm : 494\,\text{MeV}$ $\quad K^0/\overline{K}^0 : 498\,\text{MeV}$
$\eta : 549\,\text{MeV}$ $\quad\eta' : 958\,\text{MeV}$

$\rho^\pm : 770\,\text{MeV}$ $\quad\rho^0 : 770\,\text{MeV}$
$K^{*\pm} : 892\,\text{MeV}$ $\quad K^{*0}/\overline{K^*}^0 : 896\,\text{MeV}$
$\omega : 782\,\text{MeV}$ $\quad\phi : 1020\,\text{MeV}$

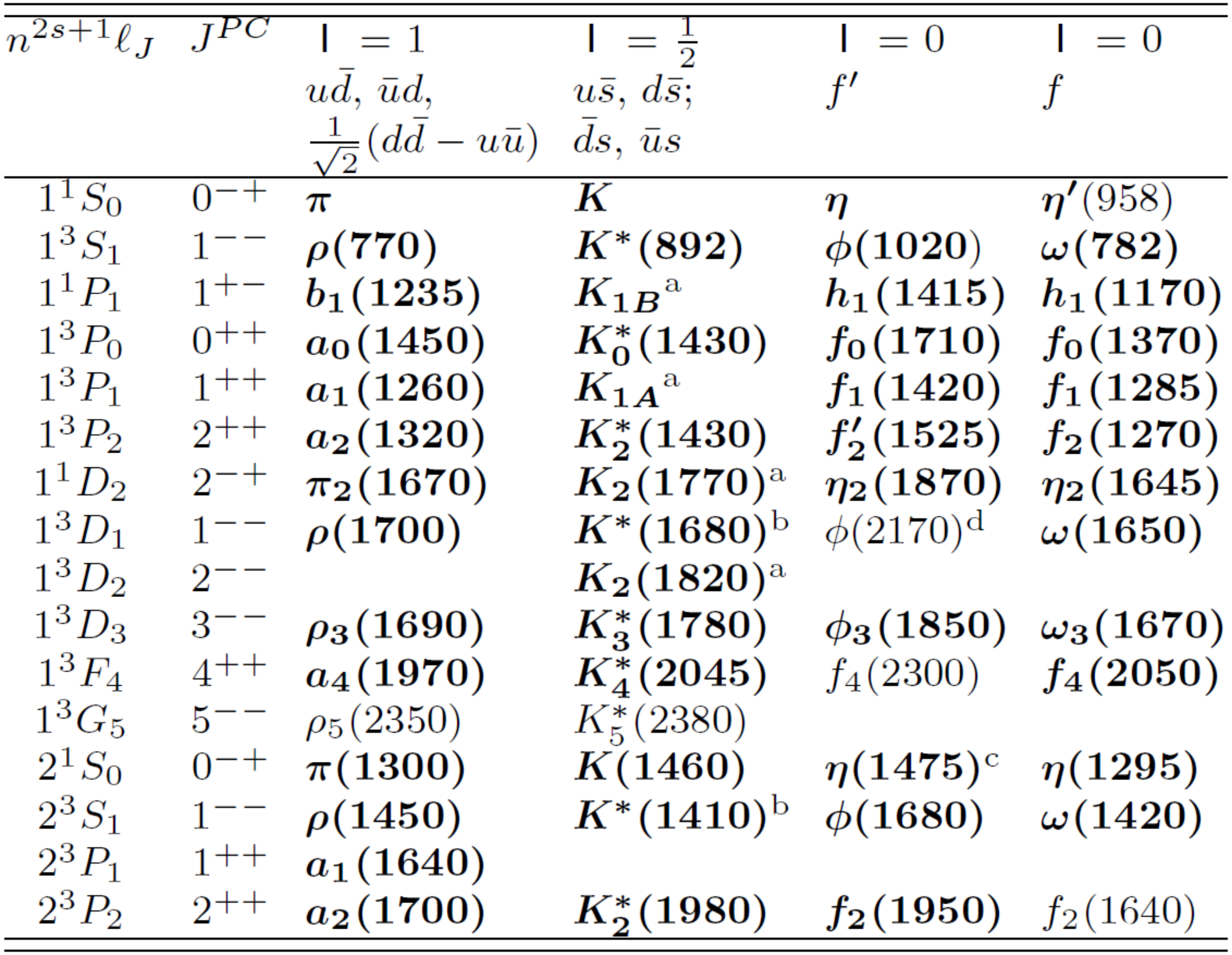
Allowed Meson States



		J^{PC}
$L = 0$	$S = 0$	0^{-+}
	$S = 1$	1^{--}
$L = 1$	$S = 0$	1^{+-}
	$S = 1$	2^{++} 1^{++} 0^{++}

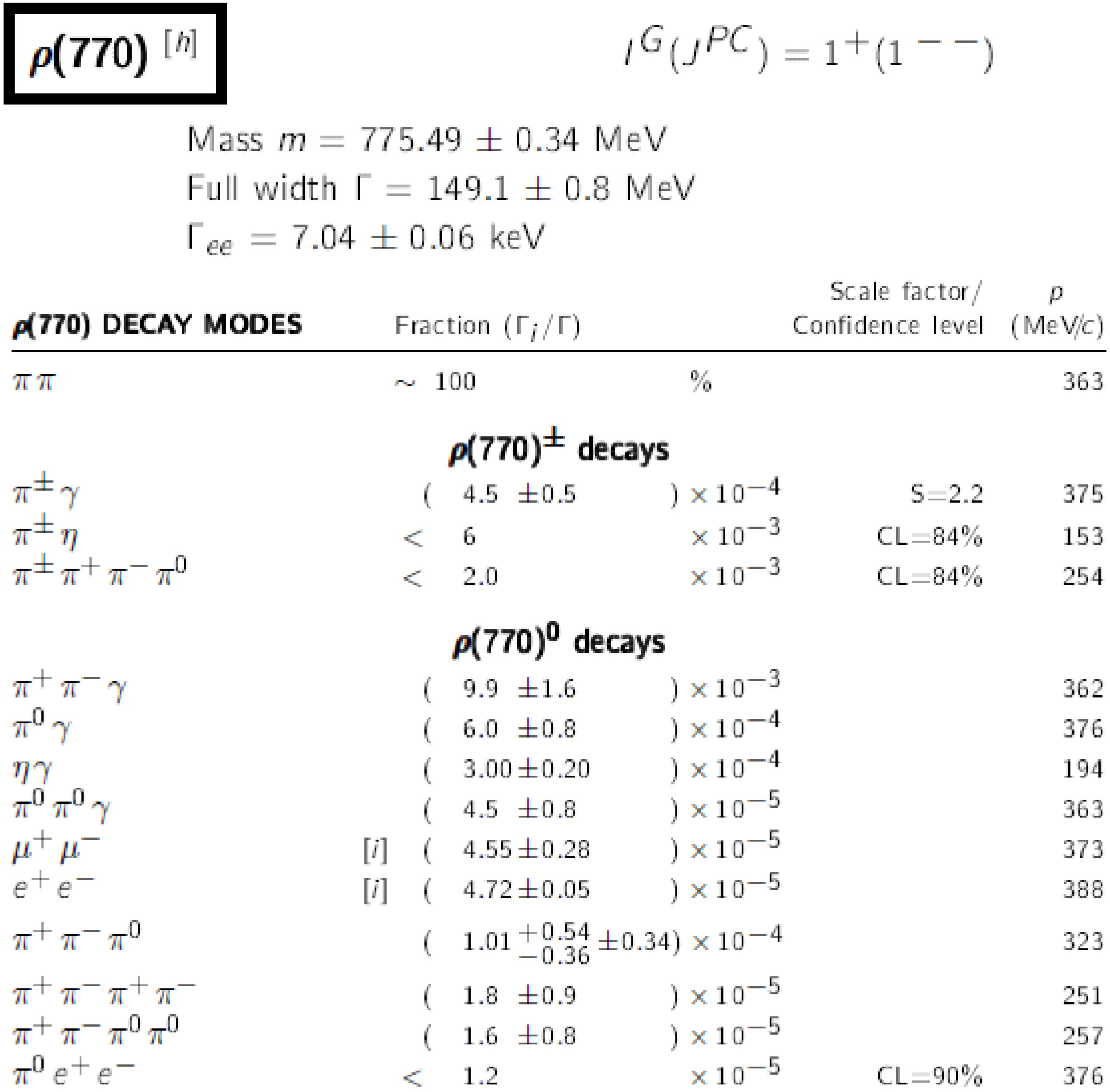
Allowed J^{PC} meson states

$J = L + S$



$n^{2s+1}\ell_J$	J^{PC}	$I = 1$ $u\bar{d},\ \bar{u}d,$ $\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$	$I = \frac{1}{2}$ $u\bar{s},\ d\bar{s};$ $\bar{d}s,\ \bar{u}s$	$I = 0$ f'	$I = 0$ f
1^1S_0	0^{-+}	π	K	η	$\eta'(958)$
1^3S_1	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$
1^1P_1	1^{+-}	$b_1(1235)$	K_{1B}[a]	$h_1(1415)$	$h_1(1170)$
1^3P_0	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$
1^3P_1	1^{++}	$a_1(1260)$	K_{1A}[a]	$f_1(1420)$	$f_1(1285)$
1^3P_2	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)$[a]	$\eta_2(1870)$	$\eta_2(1645)$
1^3D_1	1^{--}	$\rho(1700)$	$K^*(1680)$[b]	$\phi(2170)$[d]	$\omega(1650)$
1^3D_2	2^{--}		$K_2(1820)$[a]		
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^*(1780)$	$\phi_3(1850)$	$\omega_3(1670)$
1^3F_4	4^{++}	$a_4(1970)$	$K_4^*(2045)$	$f_4(2300)$	$f_4(2050)$
1^3G_5	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$		
2^1S_0	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)$[c]	$\eta(1295)$
2^3S_1	1^{--}	$\rho(1450)$	$K^*(1410)$[b]	$\phi(1680)$	$\omega(1420)$
2^3P_1	1^{++}	$a_1(1640)$			
2^3P_2	2^{++}	$a_2(1700)$	$K_2^*(1980)$	$f_2(1950)$	$f_2(1640)$

Example of a Meson Listing from PDG



$\rho(770)$ [h] $I^G(J^{PC}) = 1^+(1^{--})$

Mass $m = 775.49 \pm 0.34$ MeV

Full width $\Gamma = 149.1 \pm 0.8$ MeV

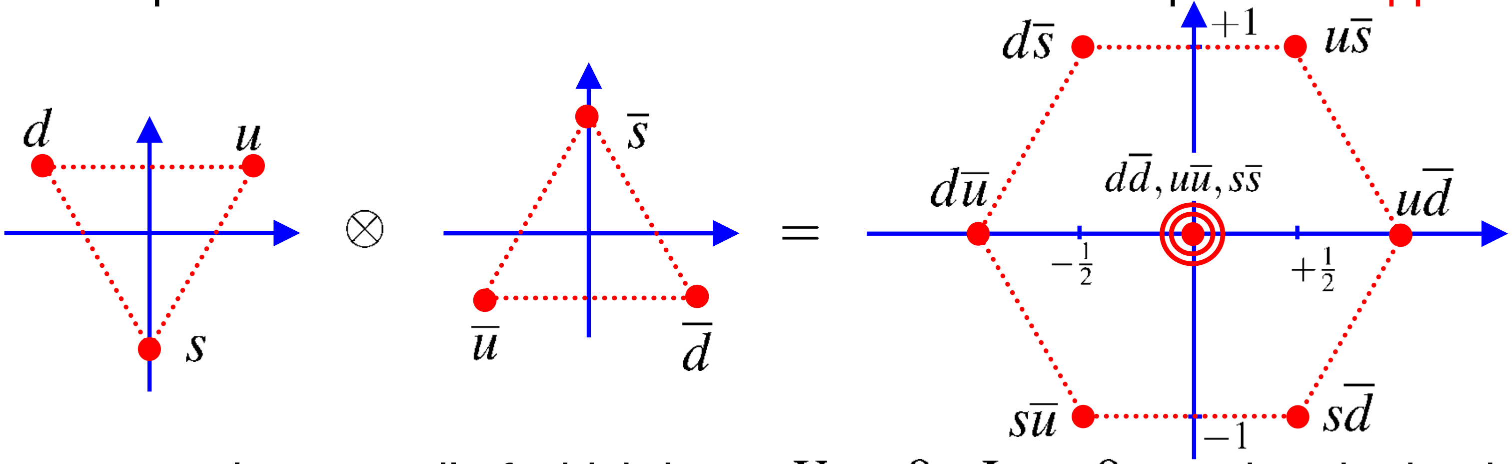
$\Gamma_{ee} = 7.04 \pm 0.06$ keV

$\rho(770)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
$\pi\pi$	~ 100 %		363
$\rho(770)^{\pm}$ decays			
$\pi^{\pm}\gamma$	$(4.5 \pm 0.5) \times 10^{-4}$	S=2.2	375
$\pi^{\pm}\eta$	$< 6 \times 10^{-3}$	CL=84%	153
$\pi^{\pm}\pi^{+}\pi^{-}\pi^{0}$	$< 2.0 \times 10^{-3}$	CL=84%	254
$\rho(770)^{0}$ decays			
$\pi^{+}\pi^{-}\gamma$	$(9.9 \pm 1.6) \times 10^{-3}$		362
$\pi^{0}\gamma$	$(6.0 \pm 0.8) \times 10^{-4}$		376
$\eta\gamma$	$(3.00 \pm 0.20) \times 10^{-4}$		194
$\pi^{0}\pi^{0}\gamma$	$(4.5 \pm 0.8) \times 10^{-5}$		363
$\mu^{+}\mu^{-}$	[i] $(4.55 \pm 0.28) \times 10^{-5}$		373
$e^{+}e^{-}$	[i] $(4.72 \pm 0.05) \times 10^{-5}$		388
$\pi^{+}\pi^{-}\pi^{0}$	$(1.01 {}^{+0.54}_{-0.36} \pm 0.34) \times 10^{-4}$		323
$\pi^{+}\pi^{-}\pi^{+}\pi^{-}$	$(1.8 \pm 0.9) \times 10^{-5}$		251
$\pi^{+}\pi^{-}\pi^{0}\pi^{0}$	$(1.6 \pm 0.8) \times 10^{-5}$		257
$\pi^{0}e^{+}e^{-}$	$< 1.2 \times 10^{-5}$	CL=90%	376

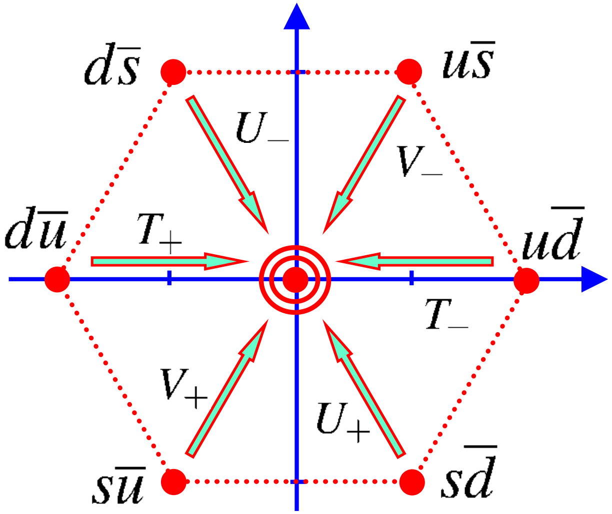
Light uds Mesons

How to form the meson states? (more rigorous approach)

Use ladder operators to construct uds mesons from the nine possible $q\bar{q}$ states.



The three central states, all of which have $Y = 0;\ I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways



$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \qquad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$

$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \qquad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$

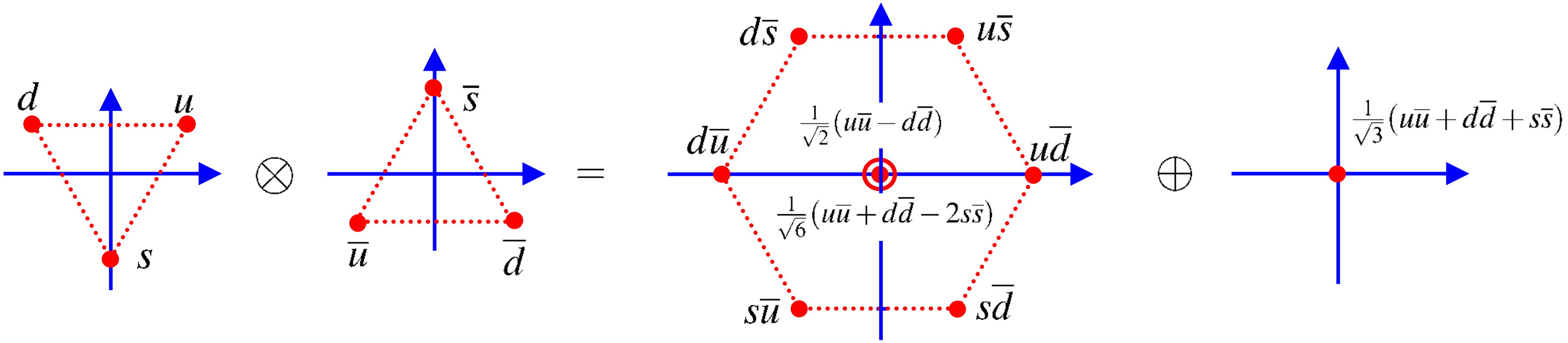
$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \qquad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$

Only two of these six states are linearly independent.

But there are three states with $Y = 0;\ I_3 = 0$

Therefore one state is not part of the same multiplet, i.e. cannot be reached with ladder operators.

Therefore the combination of a quark and anti-quark yields nine states which break down into an OCTET and a SINGLET



In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$

Using ladder operator check that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s})$ is a flavourless state,

i.e. invariant under $SU(3)_F$ flavour transformations ($U \ \Psi_3 = \Psi_3$)

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

Can compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$

TRIPLET of spin-1 states: $|1,-1\rangle, \ |1,0\rangle, \ |1,+1\rangle$

SINGLET spin-0 state: $|0,0\rangle$

These spin triplet states are connected by SU(2) ladder operators just as the meson *uds* octet states are connected by $SU(3)_F$ flavour ladder operators.
The (spin) singlet state carries no angular momentum – in this sense the $SU(3)_F$ flavour singlet is “flavourless”

Baryons: qqq States

Baryons are composed of 3 quarks bound together. Baryon states can be obtained by combining three fundamental representations of the *SU(3)*$_F$ group

$$\boxed{3\otimes 3\otimes 3 = 10_S \oplus 8_{MS} \oplus 8_{MA} \oplus 1_A}$$

Therefore there are 27 possible qqq combinations,
but what is the difference between uud, udu, or duu?
We observe only 8 baryons with spin 1/2 ħ and 10 baryons with spin 3/2 ħ.
The states must have definite symmetry under *SU(3)*$_F$ transformations.

The proton is a fermion and the wave function must be antisymmetric under the interchange of any two quarks.
Proton wave function can be decomposed as

$$\boxed{\psi_p = R(\text{space})\times\phi(\text{flavor})\times\chi(\text{spin})\times\xi(\text{color})}$$

The colour wave-function for all bound qqq states is anti-symmetric (see later).

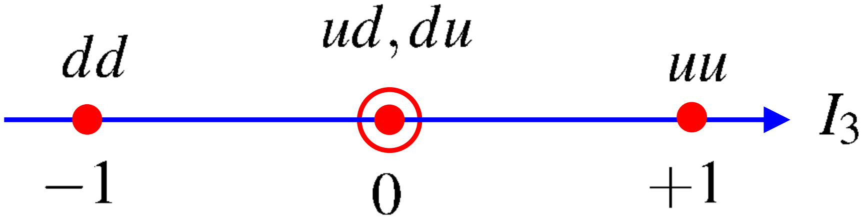
How to construct the baryon wave functions?
The $\phi(\text{flavor})\times\chi(\text{spin})$ part is symmetric under interchange of any two quarks

1. Combine two u, d quarks
2. Add the third quark (u or d)
3. Combine with spin
4. Use the *SU(3)*$_F$ ladder operators to construct the strange baryon wave functions

Combining Quarks (ud)

First combine two quarks, then add the third quark
Use the requirement that fermion wave-functions are anti-symmetric,
$\phi(\text{flavor}) \times \chi(\text{spin})$ is symmetric.

With two quarks, we have four possible combinations:



⦿ represents two states with the same value of I_3

We can immediately identify the extremes (recall I_3 is additive)

$$\boxed{uu \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,+1\rangle} \qquad \boxed{dd \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle}$$

To obtain the $|1,0\rangle$ state use the isospin lowering ladder operator

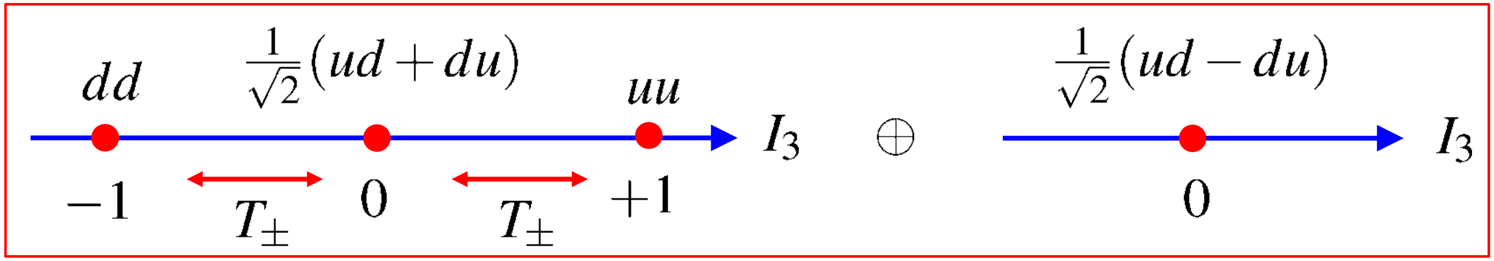
$$T_-|1,+1\rangle = \sqrt{2}|1,0\rangle = T_-(uu) = ud + du$$

➡ $$\boxed{|1,0\rangle = \frac{1}{\sqrt{2}}(ud + du)}$$

The last state, $|0,0\rangle$, can be found from orthogonality with $|1,0\rangle$

➡ $$\boxed{|0,0\rangle = \frac{1}{\sqrt{2}}(ud - du)}$$

From four possible combinations of isospin doublets we obtain a triplet of isospin 1 states and a singlet of isospin 0 state $2 \otimes 2 = 3_S \oplus 1_A$



Can move around within multiplets using ladder operators

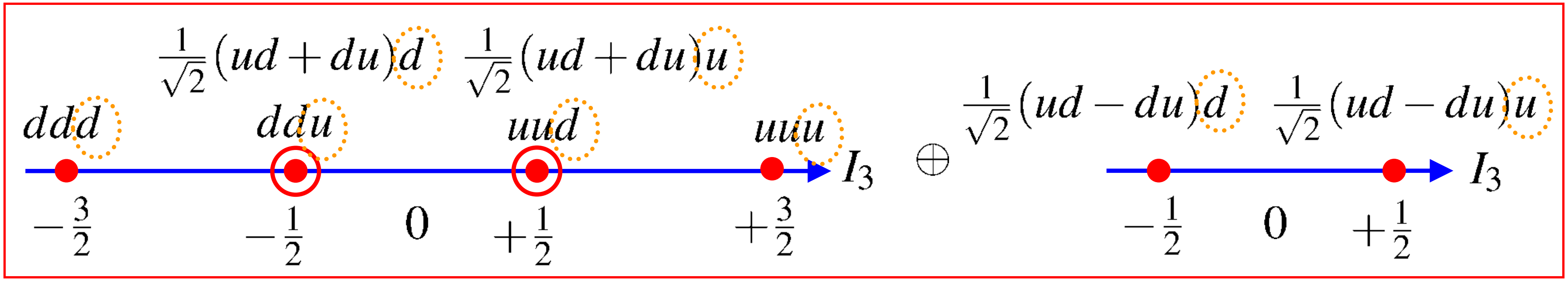
note $I_3 = 1/2\left(N_u - N_d\right)$

States with different total isospin are physically different – the isospin 1 triplet is symmetric under interchange of quarks 1 and 2, whereas the singlet is anti-symmetric.

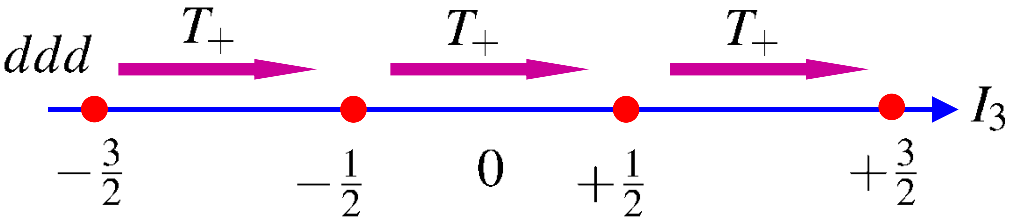
Now add the third u or d quark.

From each of the above 4 states we get two new isospin states with $I'_3 = I_3 \pm ½$.

The eight states *uuu*, *uud*, *udu*, udd, *duu*, *dud*, *ddu*, *ddd* are grouped into an isospin quadruplet (I = 3/2) and two isospin doublets (I = 1/2)



We can derive the $I = 3/2$ states from $ddd \equiv \left|\frac{3}{2},-\frac{3}{2}\right\rangle$ (or $uuu \equiv \left|\frac{3}{2},\frac{3}{2}\right\rangle$) using the ladder operators.



$$\boxed{\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = ddd}$$

$$T_+\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = T_+(ddd) = (T_+d)dd + d(T_+d)d + dd(T_+)d$$

$$\sqrt{3}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = udd + dud + ddu$$

$$\boxed{\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(udd + dud + ddu)}$$

$$T_+\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}T_+(udd + dud + ddu)$$

$$2\left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(uud + udu + uud + duu + udu + duu)$$

$$\boxed{\left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(uud + udu + duu)}$$

$$T_+\left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}T_+(uud + udu + duu)$$

$$\sqrt{3}\left|\tfrac{3}{2},+\tfrac{3}{2}\right\rangle = \frac{1}{\sqrt{3}}(uuu + uuu + uuu)$$

$$\boxed{\left|\tfrac{3}{2},+\tfrac{3}{2}\right\rangle = uuu}$$

We thus obtain 4 fully symmetric states with *I* = 3/2
which we identify with the 4 Δ resonances

$$\boxed{\Delta^{++} = uuu \qquad \Delta^{+} = \frac{1}{\sqrt{3}}\left(uud + udu + duu\right) \qquad \Delta^{+} = \frac{1}{\sqrt{3}}\left(udd + dud + ddu\right) \qquad \Delta^{-} = ddd}$$

we keep the two *I* = 1/2 states anti-symmetric under the exchange of the first two quarks,

$$p_A = \frac{1}{\sqrt{2}}\left[\left(ud - du\right)u\right] \qquad n_A = \frac{1}{\sqrt{2}}\left[\left(ud - du\right)d\right]$$

and obtain the remaining two *I* = 1/2 states symmetric for the exchange of the first two quarks by orthogonality

$$p_S = \frac{1}{\sqrt{6}}\left[2uud - \left(ud + du\right)u\right] \qquad n_S = -\frac{1}{\sqrt{6}}\left[2ddu - \left(ud + du\right)d\right]$$

In summary we decomposed the 2 ⊗ 2 ⊗ 2 isospin representation in representations with definite symmetry properties under the interchange of any two quarks

The eight states *uuu*, *uud*, *udu*, udd, *duu*, *dud*, *ddu*, *ddd*
are grouped into an isospin quadruplet and two isospin doublets

$$2 \otimes 2 \otimes 2 = (3_S \oplus 1_A) \otimes 2 = (2 \otimes 3) \oplus (2 \otimes 1) = 4_S \oplus 2_{MA} \oplus 2_{MS}$$

Different multiplets have different symmetry properties

$$\left.\begin{aligned}
|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= uuu \\
|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud+udu+duu) \\
|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(ddu+dud+udd) \\
|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= ddd
\end{aligned}\right\} \boxed{\mathbf{S}}$$

A quadruplet of states which are symmetric under the interchange of any two quarks

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\tfrac{1}{\sqrt{6}}(2ddu-udd-dud) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2uud-udu-duu)
\end{aligned}\right\} \boxed{\mathbf{M_S}}$$

A doublet with
mixed symmetry.
Symmetric for 1 ⟷ 2

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(udd-dud) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(udu-duu)
\end{aligned}\right\} \boxed{\mathbf{M_A}}$$

A doublet with
mixed symmetry.
Anti-symmetric for 1 ⟷ 2

Mixed symmetry states have definite symmetry under interchange of the first two quarks 1 ↔ 2 , but not for quarks 1 ↔ 3 and 2 ↔ 3 .

To form the baryon’s wave functions we have to add the spin of the quarks.

Adding Spin

Can apply exactly the same mathematics to determine the possible spin wave-functions for a combination of 3 spin ½ particles

$$2\otimes 2\otimes 2=\left(3_S\oplus 1_A\right)\otimes 2=4_S\oplus 2_{MS}\oplus 2_{MA}$$

$$\left.\begin{aligned}
|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= \uparrow\uparrow\uparrow \\
|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow+\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow) \\
|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow+\downarrow\uparrow\downarrow+\uparrow\downarrow\downarrow) \\
|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= \downarrow\downarrow\downarrow
\end{aligned}\right\} \boxed{S}$$

A quadruplet of states which are symmetric under the interchange of any two quarks

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow-\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)
\end{aligned}\right\} \boxed{M_S}$$

Mixed symmetry.
Symmetric for 1 ⟷ 2

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)
\end{aligned}\right\} \boxed{M_A}$$

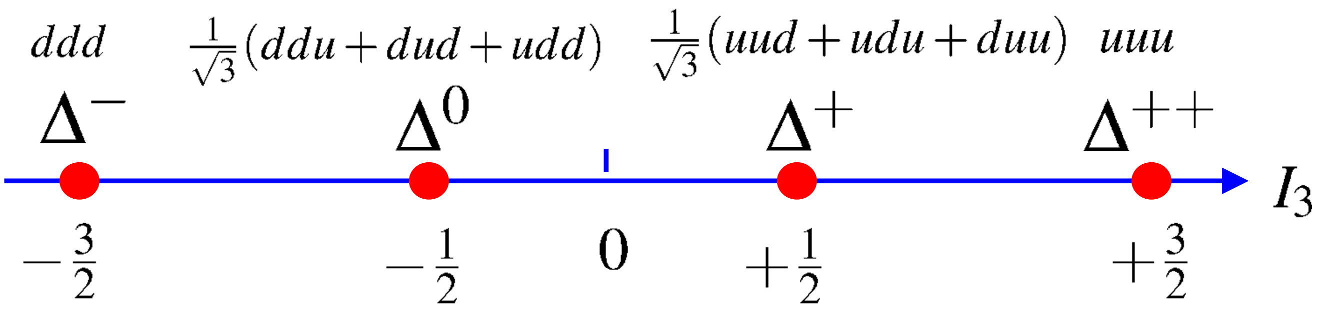
Mixed symmetry.
Anti-symmetric for 1 ⟷ 2

Now we can form total wave-functions for combination of three quarks

Baryon Wave-Functions (ud quarks only)

Two ways to form a totally symmetric wave function from spin and isospin states:

A combine totally symmetric spin and isospin wave functions $\boxed{\phi(S)\chi(S)}$



Spin 3/2
Isospin 3/2

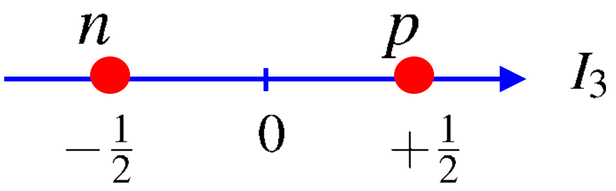
B combine mixed symmetry spin and mixed symmetry isospin states

both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under interchange of quarks 1 $\leftrightarrow$ 2
not sufficient, these combinations have no definite symmetry under 1 $\leftrightarrow$ 3, …

however, the (normalised) linear combination

$$\boxed{\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S)+\frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)}$$

is totally symmetric (i.e. symmetric under 1 $\leftrightarrow$ 2, 1 $\leftrightarrow$ 3, 2 $\leftrightarrow$ 3)



Spin 1/2
Isospin 1/2

while the orthogonal combination

$$\boxed{\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S)+\frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)}$$

is totally anti-symmetric (i.e. anti-symmetric under 1 ↔ 2, 1 ↔ 3, 2 ↔ 3)
(in principle can build p, and n wavefunctions with no need of color, but not for other baryon states)

The spin-up proton wave-function is therefore

$$\left|p\uparrow\right\rangle = \sqrt{1/2}\left(\varphi_S \otimes \chi_S + \varphi_A \otimes \chi_A\right)$$

$$= \sqrt{1/72}\left(2uud-udu-duu\right)\left(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow\right)+\sqrt{1/8}\left(udu-duu\right)\left(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow\right)$$

$$= \sqrt{1/18}\left[2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \text{permutations}\right]$$

$$\Rightarrow \left|p\uparrow\right\rangle = \sqrt{1/18}(2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow +$$
$$2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow +$$
$$2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\downarrow \quad)$$

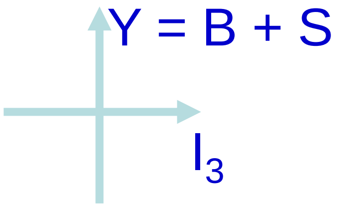
Not always necessary to use the fully symmetrised proton wave function,
e.g. the first 3 terms are sufficient for calculating the proton magnetic moment.

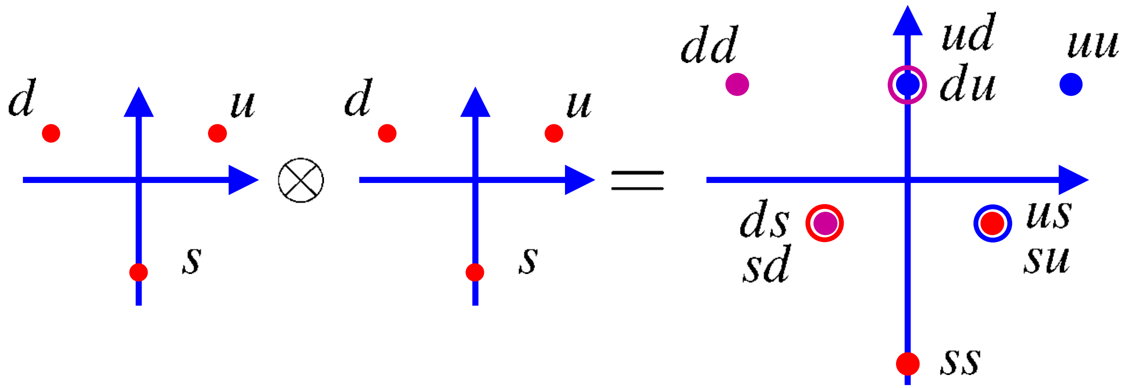
Now use U and V ladder operators to introduce the s quark.

Combining uds Quarks into Baryons

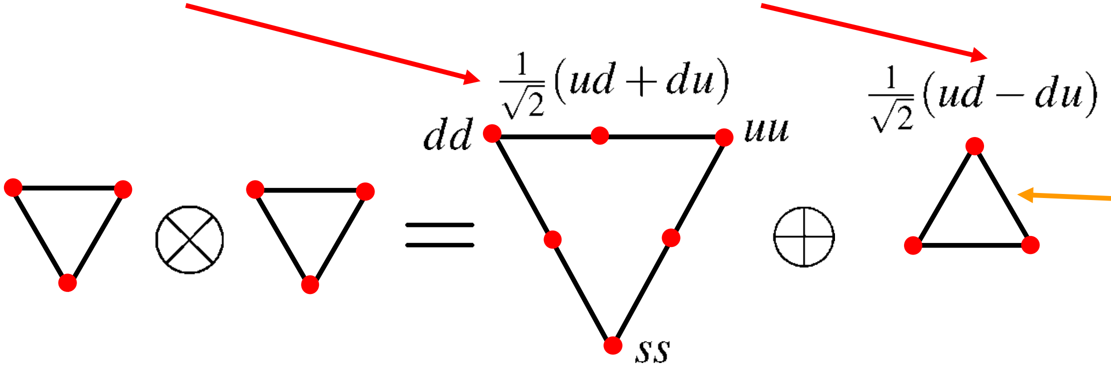
Constructing baryon states is a fairly elaborate process, see the derivation of the proton wave function.
Concentrate on multiplet structure rather than deriving all the wave-functions.

First combine two quarks:





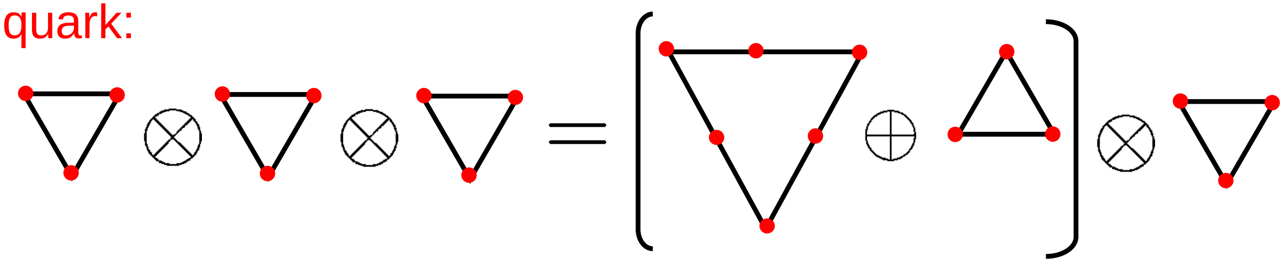
This yields a symmetric sextet and anti-symmetric triplet: $3 \otimes 3 = 6 \oplus \overline{3}$



Same “pattern” as the anti-quark representation

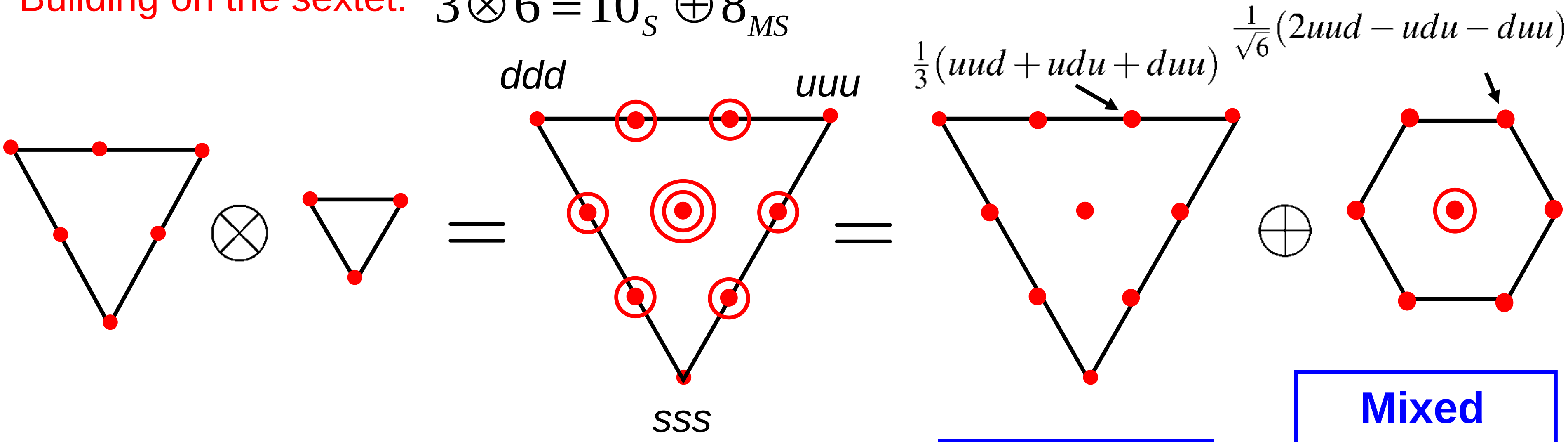
SYMMETRIC **ANTI-SYMMETRIC**

Add the third quark:



Best considered in two parts, building on the sextet and triplet. Again concentrate on the multiplet structure (for the wave functions refer to the discussion of proton wave function).

Building on the sextet: $3\otimes 6 = 10_S \oplus 8_{MS}$

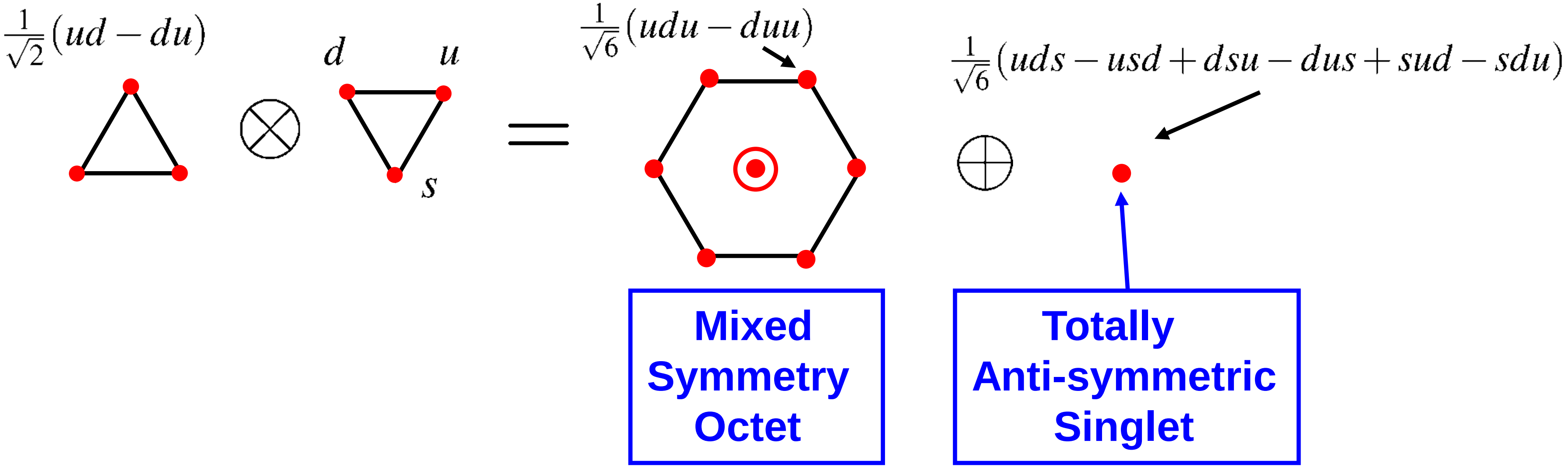


Symmetric Decuplet

Mixed Symmetry Octet

Building on the triplet: $\bar{3}\otimes 3 = 8\oplus 1$

Just as in the case of *uds* mesons we are combining $\bar{3}\otimes 3$ and again obtain an octet and a singlet



$\frac{1}{\sqrt{2}}(ud-du) \otimes \quad = \quad \oplus$

$\frac{1}{\sqrt{6}}(udu-duu)$

$\frac{1}{\sqrt{6}}(uds-usd+dsu-dus+sud-sdu)$

Mixed Symmetry Octet

Totally Anti-symmetric Singlet

Can verify the wave function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds-usd+dsu-dus+sud-sdu)$ is a singlet by using ladder operators, e.g.

$$T_{+}\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus-usu+usu-uus+suu-suu) = 0$$

In summary, the combination of three *uds* quarks decomposes into

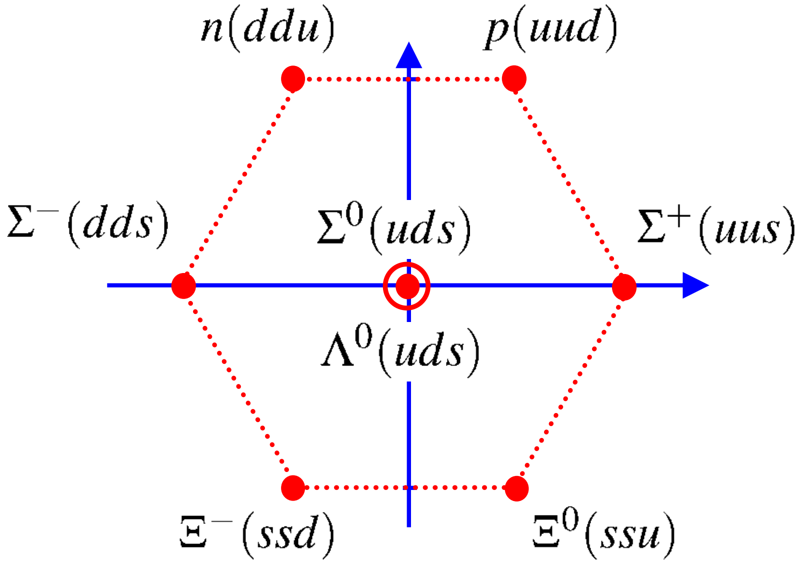
$$\boxed{3\otimes 3\otimes 3 = 3\otimes(6\oplus\bar{3}) = 10_S \oplus 8_{MS} \oplus 8_{MA} \oplus 1_A}$$

Baryon Octet

The spin 1/2 octet is formed from mixed symmetry flavour and mixed symmetry spin wave functions

$$\boxed{\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S)+\frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)}$$

BARYON OCTET (L=0, S=1/2, J=1/2, P= +1)



Mass in MeV

N(939) I = 1/2

Σ(1193) I = 1
Λ(1116) I = 0

Ξ(1318) I = 1/2

We cannot form a totally symmetric wave function based on the anti-symmetric flavour singlet as there are no totally anti-symmetric spin wave functions for 3 quarks.

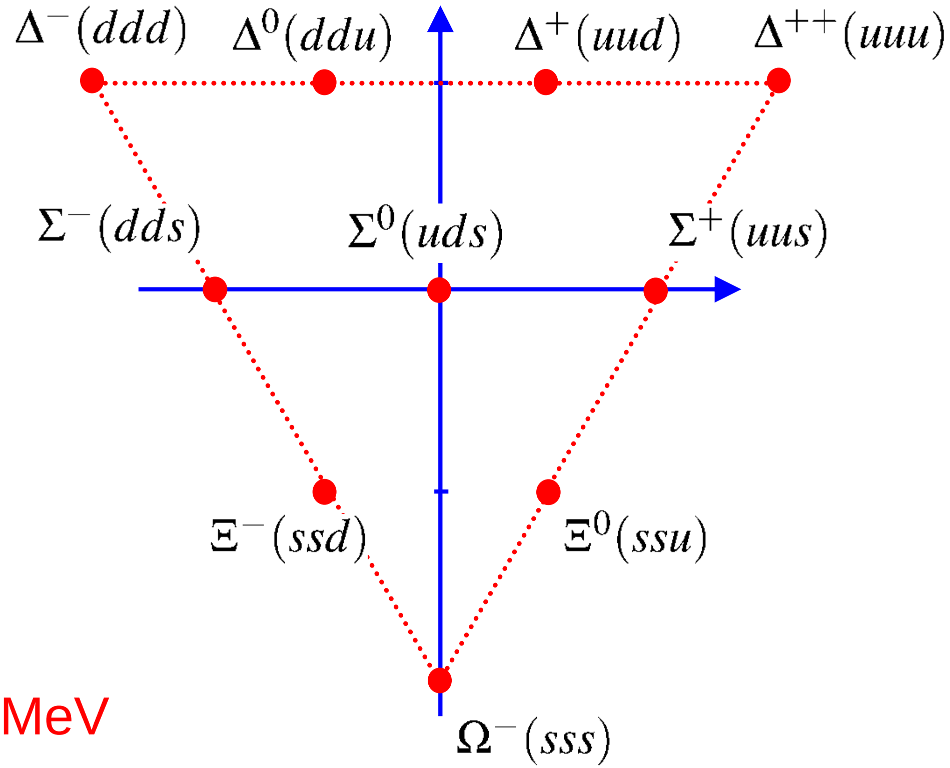
Baryon Decuplet

The baryon states (L=0) are
the spin 3/2 decuplet of symmetric flavour and symmetric spin wave-functions

$$\boxed{\phi(S)\chi(S)}$$

BARYON DECUPLET (L=0, S=3/2, J=3/2, P= +1)

Mass in MeV



Δ(1232) I = 3/2

Σ*(1384) I = 1

Ξ* (1530) I = 1/2

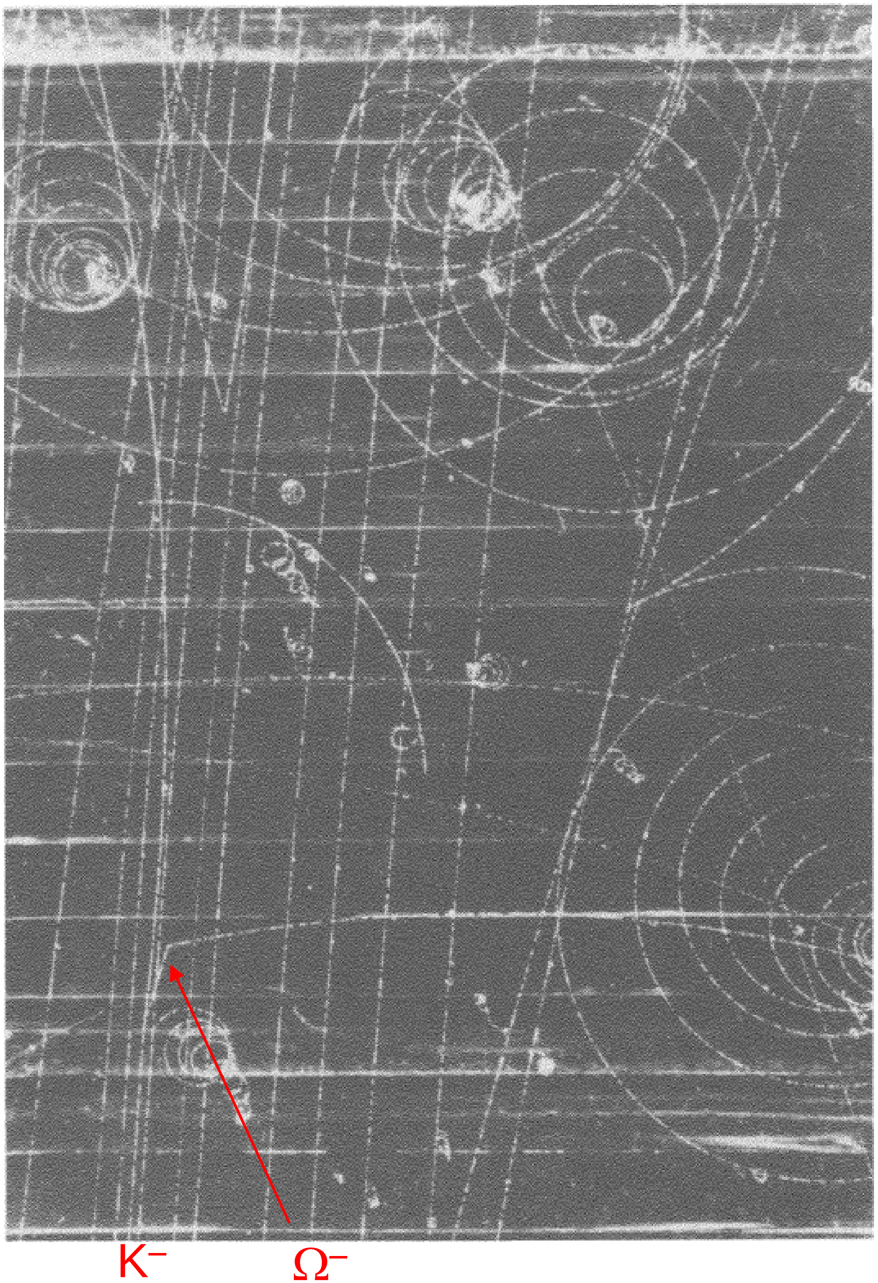
Ω(1672) I = 0

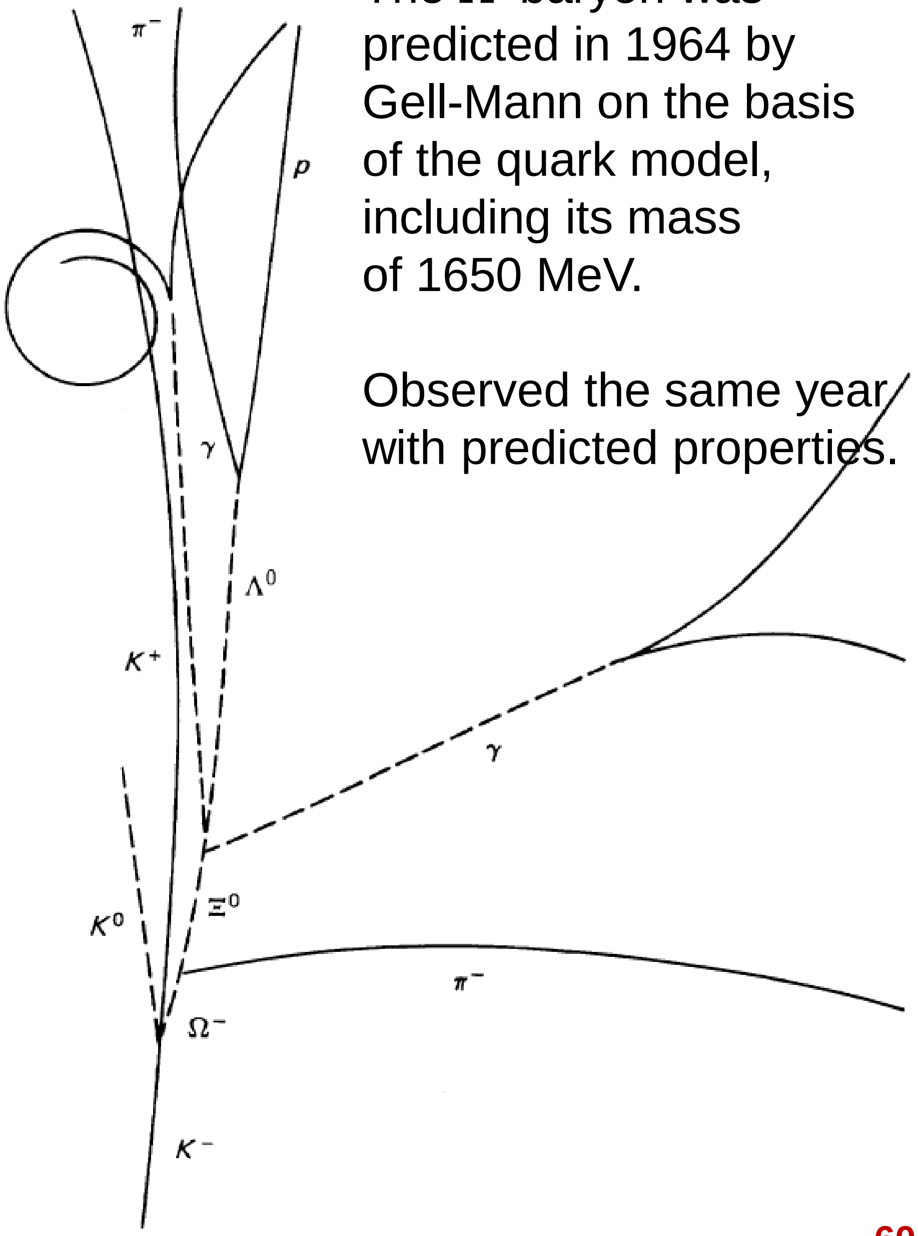
fully symmetric
qqq combinations

prediction:
Ω^- around 1650 MeV

If $SU(3)_F$ flavour were an exact symmetry
all masses would be the same (broken symmetry)

Discovery of Ω^-



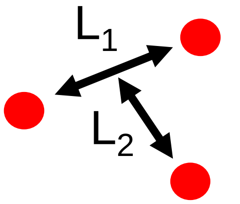


The Ω^- baryon was predicted in 1964 by Gell-Mann on the basis of the quark model, including its mass of 1650 MeV.

Observed the same year with predicted properties.

Excitation Spectrum for Baryons

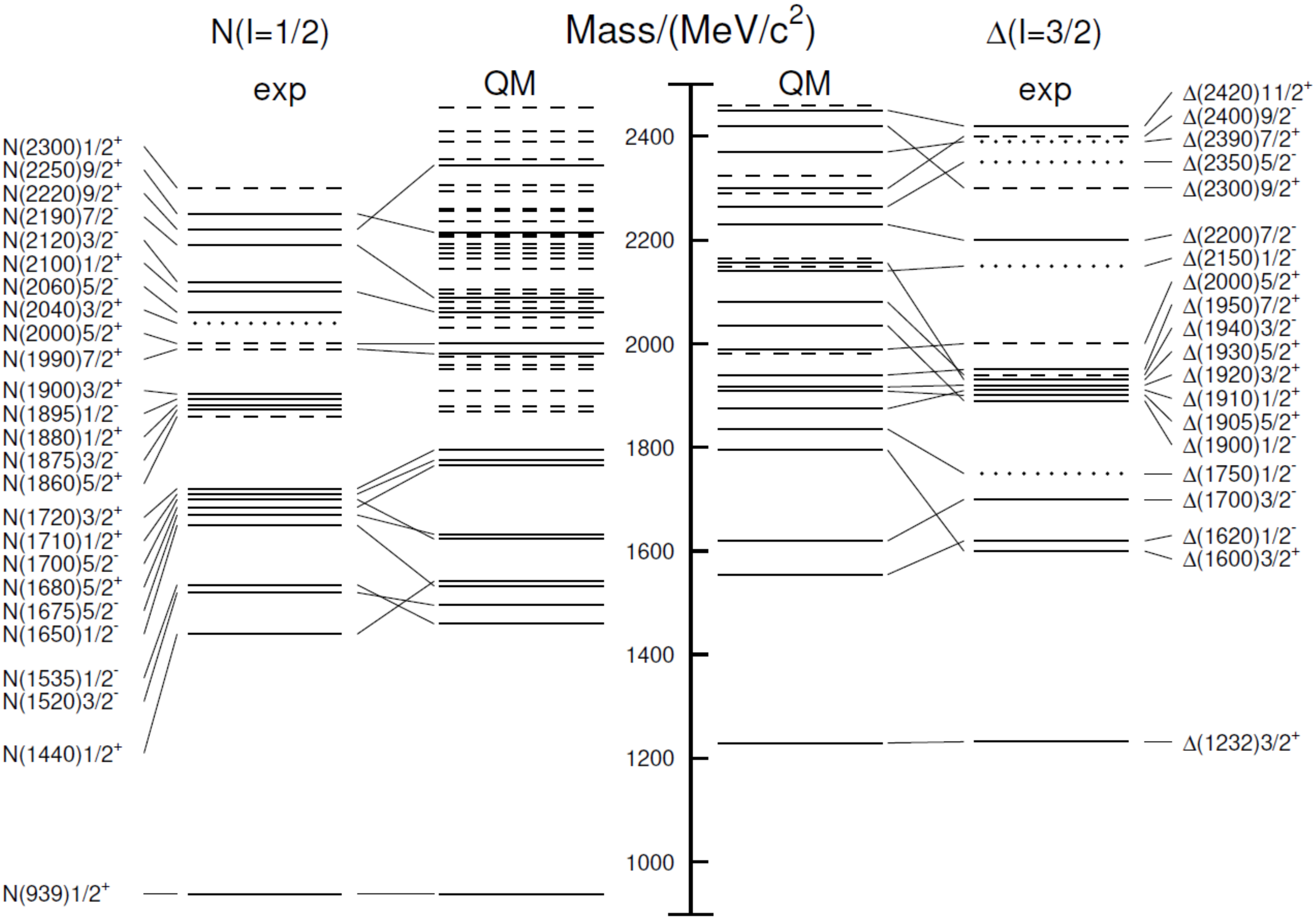
The nucleons can also be created in an excited states of higher angular momentum (resonances → they decay quickly to the ground state)



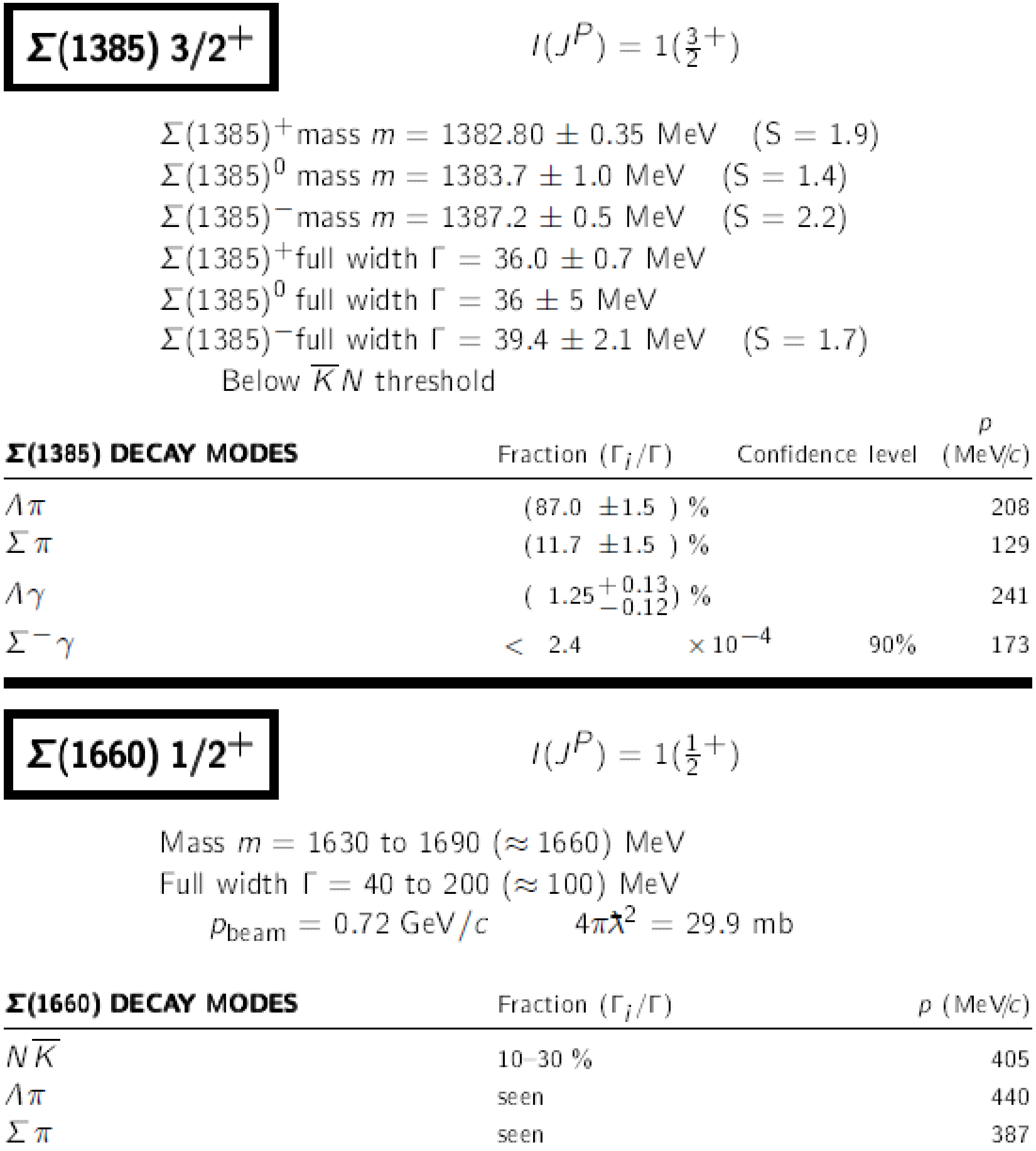
$L = L_1 + L_2$

$J = S + L$

$P = (-1)^L$



Example of Baryon Listings from PDG



$\Sigma(1385)$ 3/2$^+$ $I(J^P) = 1(\frac{3}{2}^+)$

$\Sigma(1385)^+$ mass $m = 1382.80 \pm 0.35$ MeV (S = 1.9)
$\Sigma(1385)^0$ mass $m = 1383.7 \pm 1.0$ MeV (S = 1.4)
$\Sigma(1385)^-$ mass $m = 1387.2 \pm 0.5$ MeV (S = 2.2)
$\Sigma(1385)^+$ full width $\Gamma = 36.0 \pm 0.7$ MeV
$\Sigma(1385)^0$ full width $\Gamma = 36 \pm 5$ MeV
$\Sigma(1385)^-$ full width $\Gamma = 39.4 \pm 2.1$ MeV (S = 1.7)
Below $\overline{K}N$ threshold

Σ(1385) DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\Lambda\pi$	(87.0 ± 1.5) %		208
$\Sigma\pi$	(11.7 ± 1.5) %		129
$\Lambda\gamma$	$(1.25^{+0.13}_{-0.12})$ %		241
$\Sigma^-\gamma$	$< 2.4 \times 10^{-4}$	90%	173

$\Sigma(1660)$ 1/2$^+$ $I(J^P) = 1(\frac{1}{2}^+)$

Mass $m = 1630$ to 1690 (≈ 1660) MeV
Full width $\Gamma = 40$ to 200 (≈ 100) MeV
$p_{\text{beam}} = 0.72$ GeV$/c$ $4\pi\lambdabar^2 = 29.9$ mb

Σ(1660) DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$N\overline{K}$	10–30 %	405
$\Lambda\pi$	seen	440
$\Sigma\pi$	seen	387

Exotica

In principle, states as

$q\bar{q}q\bar{q}$ - tetraquarks (e.g. $f_0(500) = \bar{u}\bar{d}ud$)

$qqq\bar{q}q$ - pentaquarks (e.g. $\Theta^+(1540) = udud\bar{s}$)

$qqqqqq$ - *hexaquarks (or dibaryons)*

$q\bar{q}g$ - *hybrid mesons*

are allowed.
We can form a color singlet, while combinations as *qq* or *qqqq* are forbidden (no singlet!)

Some such states have been observed, however not firmly established.
Are they genuine new meson states, or just meson molecules?

States like *gg* or *ggg* are even predicted by QCD - glueballs.

More exotic states observed involving heavy quarks (i.e. charm)

Issues in the Quark Model

Are quarks real?
If so, why we did never observe free quarks?
i.e. the particles of the fundamental representation of $SU(3)_F$
⇒ confinement

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

If so, how are the quarks (partons) distributed inside the nucleon?
and how do they interact?

Inside the nucleon, quarks behave as almost free → quark – parton model
⇒ asymptotic freedom

Hadron spectroscopy: spin-statistics problem
The fully symmetric wave function under flavor and spin is problematic,
baryons are fermions → wave function must be antisymmetric under interchange of any two quarks.

$$\left|p\uparrow\right\rangle = \frac{1}{\sqrt{18}}\left(2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \text{permutations}\right)$$

$$\left|\Delta^{++}\uparrow\right\rangle = u\uparrow u\uparrow u\uparrow \qquad \text{or} \qquad \left|\Omega^{-}\uparrow\right\rangle = s\uparrow s\uparrow s\uparrow$$

symmetric under space and spin *rotations*
3 identical spin-1/2 quarks with same quantum numbers in ground state
(violates Pauli exclusion principle!)

Introduction of color

SOLUTION add a new quantum number – the color – to distinguish the three quarks

and require that the wave function is totally anti-symmetric w.r.t. color

color obeys *SU(3)*$_C$ color symmetry and comes in three “charges”: red, green blue

the *SU(3)*$_C$ color symmetry is exact

color singlet: $\xi_{color} = \varepsilon_{abc} q_a q_b q_c = \sqrt{1/6}$ (RGB – RBG + BRG – BGR + GBR - GRB)

In general, the baryon wave function is decomposed as

$$\boxed{\underset{\text{anti-symmetric}}{\psi_p} = \underbrace{R(\text{space})\times\phi(\text{flavor})\times\chi(\text{spin})}_{\text{symmetric}}\times\underset{\text{anti-symmetric}}{\xi(\text{color})}}$$

and the anti-symmetry of the wave function is recovered (space × flavor × spin × color).

example: $\boxed{\left|\Omega^- \uparrow\right\rangle = s_R\uparrow s_G\uparrow s_B\uparrow}$

(qqq) color singlet √1/6 (RGB – RBG + BRG – BGR + GBR - GRB)
fully anti-symmetric ⇒ maximize attraction between quarks

All hadrons are color singlets. At this point, the color plays no dynamical role.

Observation of Quark Jets

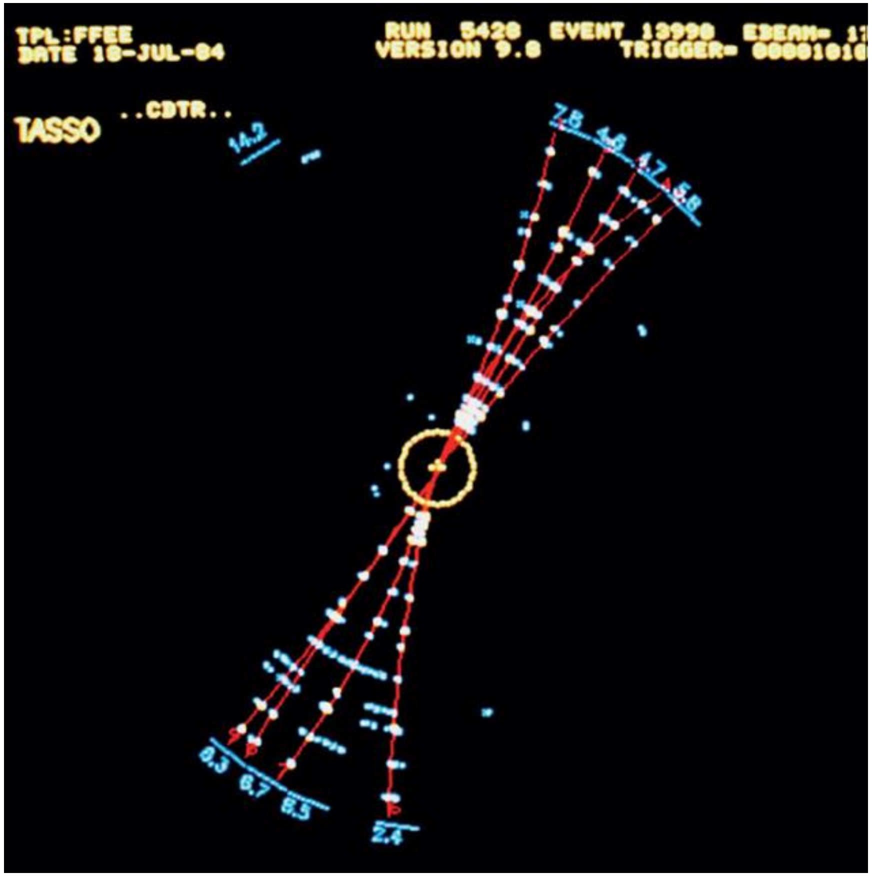
Jet = collimated spray of hadrons from quark or gluon production

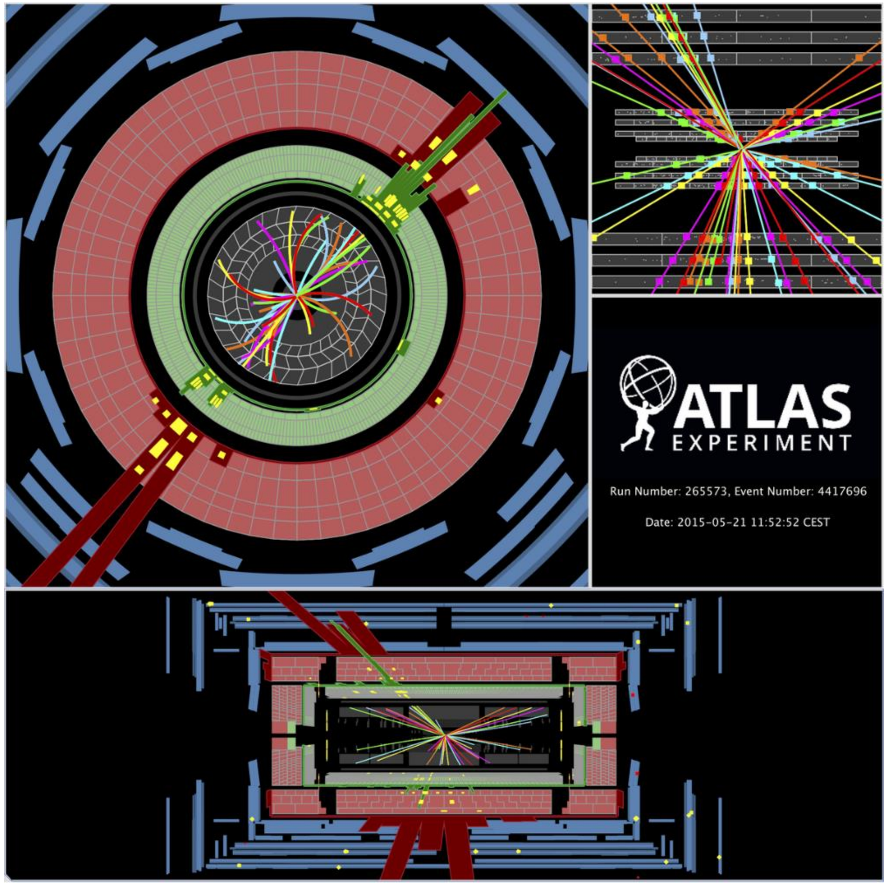
$$e^+\ e^- \rightarrow \text{jet}_1 + \text{jet}_2$$

$$\left(e^+\ e^- \rightarrow q\ \bar{q}\right)$$

$$p\ p \rightarrow \text{jet}_1 + \text{jet}_2 + X$$

$$\left(q\ \bar{q} \rightarrow q\ \bar{q}\right)$$

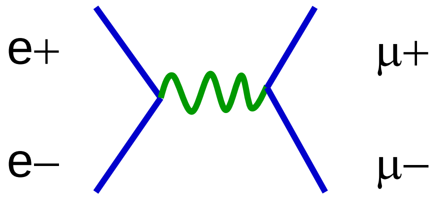




To see jets, need quarks with sufficient energy.

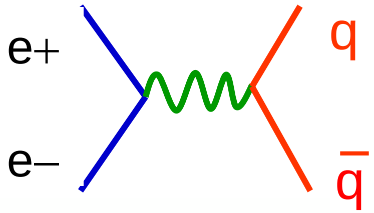
Angular Distribution of Jets

in

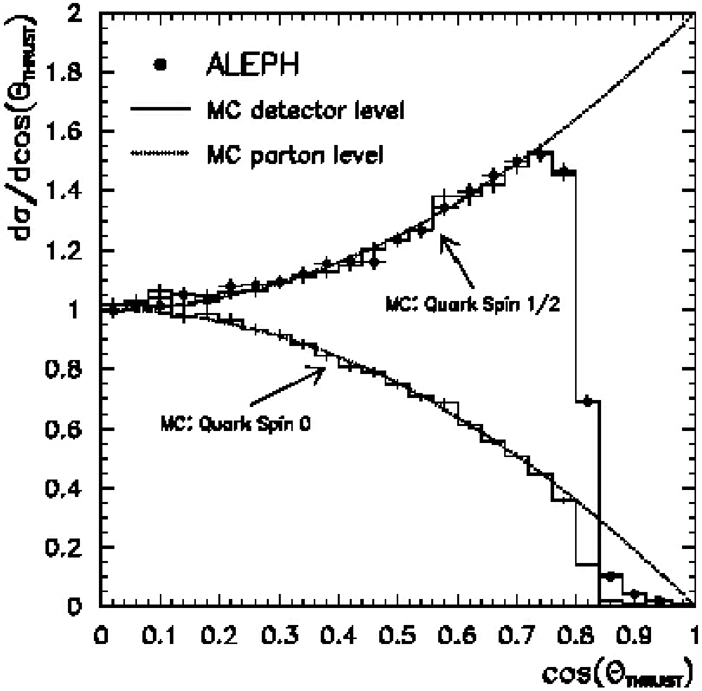


$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{8E^2}\left(1+\cos^2\theta\right)}$$

in



$$\boxed{\frac{d\sigma}{d\cos\theta} = \sum_q 3\frac{\pi Q_q^2\alpha^2}{8E^2}\left(1+\cos^2\theta\right)}$$



extra factors:
3 for color, and Q_q for quark charges

Angular distribution sensitive to spin.

Quarks have spin ½.
(historically determined via DIS)

Magnetic Moments of Baryons

One recovers the hadron properties by adding up the quark properties

charge operator $$\boxed{Q = \sum Q_i}$$

proton charge $$Q_p = \langle p\uparrow|Q|p\uparrow\rangle = \langle p\downarrow|Q|p\downarrow\rangle = 1$$

neutron charge $$Q_n = \langle n\uparrow|Q|n\uparrow\rangle = \langle n\downarrow|Q|n\downarrow\rangle = 0$$

magnetic moment operator $$\boxed{\vec{\mu} = \sum_i \vec{\mu}_i = \sum_i Q_i \frac{e}{m_i c}\vec{S}_i}$$

assuming quarks are Dirac particles with m_i

magnetic moment of quarks $$\mu_u = \frac{2}{3}\frac{e\hbar}{2m_u c} \quad \mu_d = -\frac{1}{3}\frac{e\hbar}{2m_d c} \quad \mu_s = -\frac{1}{3}\frac{e\hbar}{2m_s c}$$

magnetic moment of baryons $$\boxed{\mu_B = \langle B\uparrow|\left(\vec{\mu}_1 + \vec{\mu}_2 + \vec{\mu}_3\right)_z|B\uparrow\rangle = \frac{2}{\hbar}\sum_{i=1}^{3}\langle B\uparrow|\mu_i S_{i,z}|B\uparrow\rangle}$$

proton $$\mu_p = \sum_{i=1}^{3}\langle p\uparrow|\mu_i|p\uparrow\rangle = \frac{1}{3}(4\mu_u - \mu_d)$$

neutron $$\mu_n = \sum_{i=1}^{3}\langle n\uparrow|\mu_i|n\uparrow\rangle = \frac{1}{3}(4\mu_d - \mu_u)$$

lambda $$\mu_\Lambda = \qquad\qquad = \mu_s$$

can solve this system of equations to extract quark masses:
m_u ~ m_d ~ 340 MeV, m_s ~ 480 MeV

Note: the states are normalized $\langle u\uparrow | u\uparrow\rangle = 1$ and orthogonal $\langle u\uparrow | u\downarrow\rangle = 0$

proton wavefunction $| p\uparrow\rangle = \sqrt{1/18}\left[2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \text{perm.}\right]$

Need to calculate the first three terms and multiply by 3 (permuatations)

First calculate

$$\left(\mu_1 S_{1,z} + \mu_2 S_{2,z} + \mu_3 S_{3,z}\right)|u\uparrow u\uparrow d\downarrow\rangle = \left[\mu_u \frac{\hbar}{2} + \mu_u \frac{\hbar}{2} - \mu_d \frac{\hbar}{2}\right]|u\uparrow u\uparrow d\downarrow\rangle$$

then the first term contributes

$$\left(\frac{2}{\sqrt{18}}\right)^2 \frac{2}{\hbar} \sum_{i=1}^{3} \langle u\uparrow u\uparrow d\downarrow | \mu_i S_{i,z} | u\uparrow u\uparrow d\downarrow\rangle = \frac{2}{9}(2\mu_u - \mu_d)$$

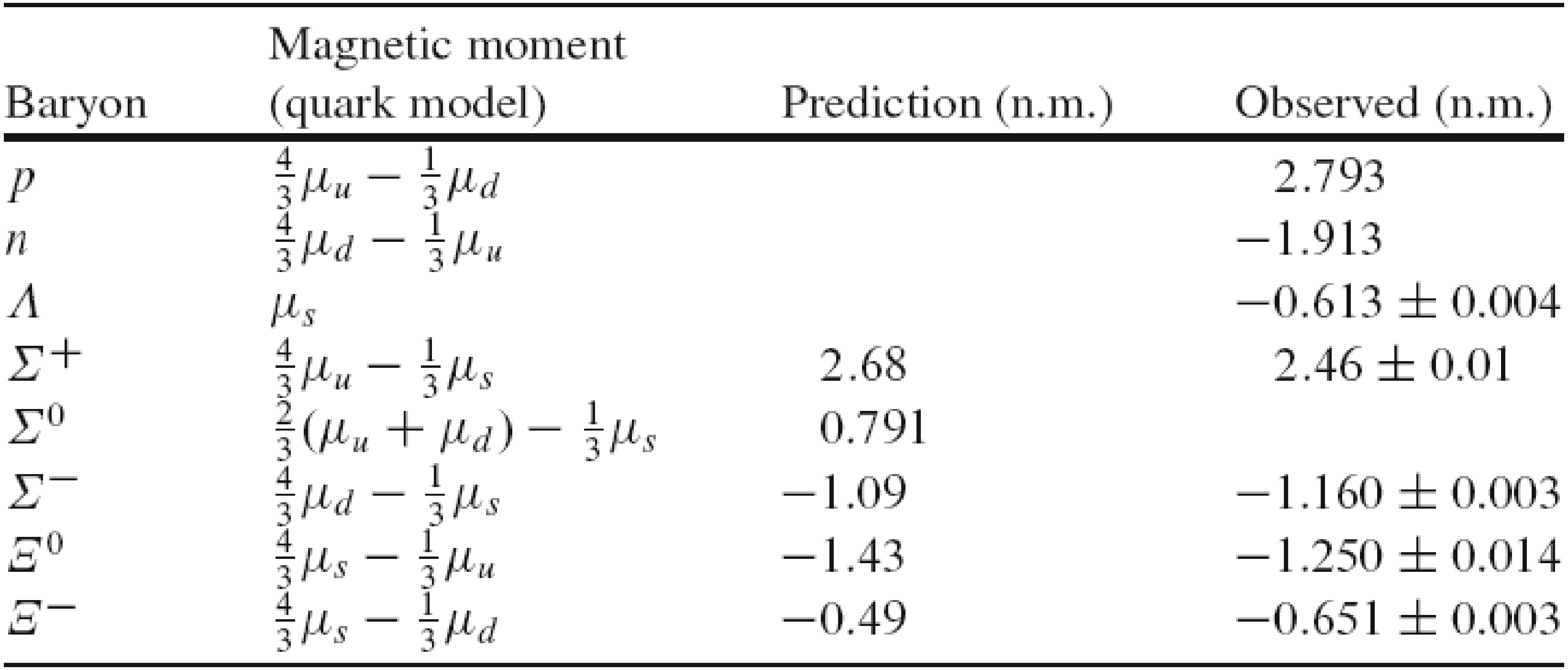
and the second and third $\frac{1}{18}\mu_d$ and $\frac{1}{18}\mu_d$

Finally $\mu_p = 3\left[\frac{2}{9}(2\mu_u - \mu_d) + \frac{1}{18}\mu_d + \frac{1}{18}\mu_d\right] = \frac{1}{3}(4\mu_u - \mu_d)$

and using isospin symmetry $\mu_n = \frac{1}{3}(4\mu_d - \mu_u)$

Magnetic Moments

input



Baryon	Magnetic moment (quark model)	Prediction (n.m.)	Observed (n.m.)
p	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$		2.793
n	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$		-1.913
Λ	μ_s		-0.613 ± 0.004
Σ^+	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.68	2.46 ± 0.01
Σ^0	$\frac{2}{3}(\mu_u + \mu_d) - \frac{1}{3}\mu_s$	0.791	
Σ^-	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	-1.09	-1.160 ± 0.003
Ξ^0	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	-1.43	-1.250 ± 0.014
Ξ^-	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	-0.49	-0.651 ± 0.003

good agreement with measurements!

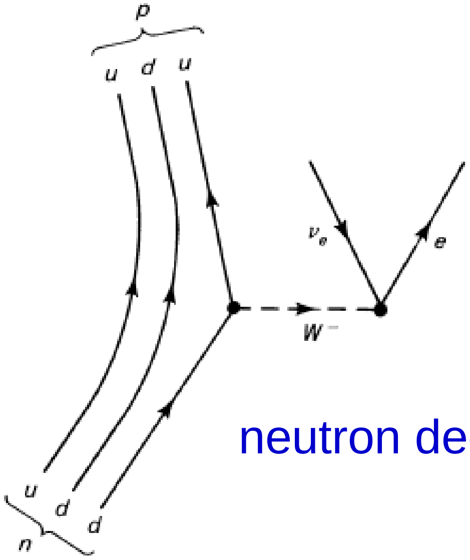
In the limit of exact isospin symmetry ($m_u = m_d$) $\mu_u = -2\mu_d$ and the ratio

$$\boxed{\frac{\mu_n}{\mu_p}(\mathrm{QM}) = -\frac{2}{3} = -0.66666 \qquad \frac{\mu_n}{\mu_p}(\exp) = -0.68497945 \pm 0.00000058}$$

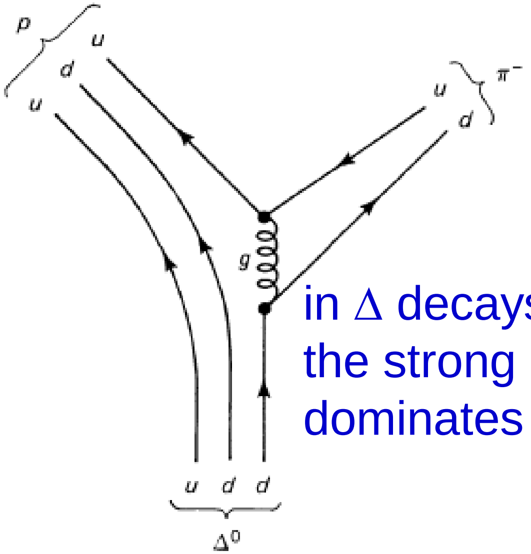
Note: the prediction -2/3 comes from the nucleon wave-function symmetric under flavor and spin

an antisymmetric nucleon wave-function under flavor and spin would predict 0.5

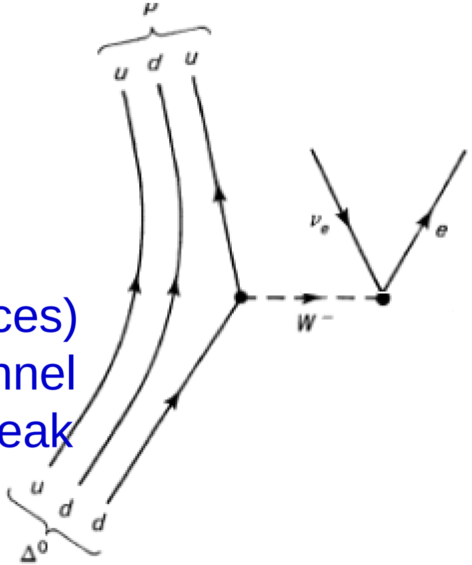
Some Baryon Decays

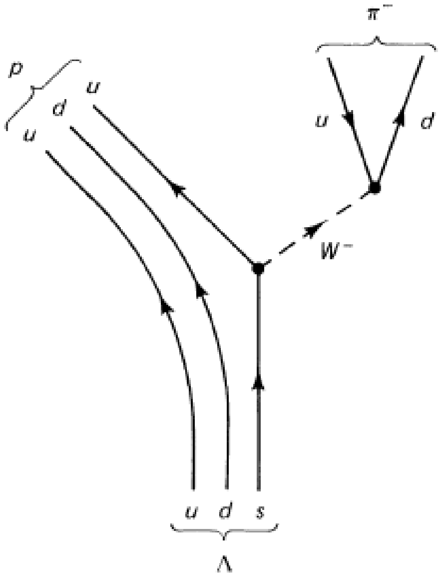


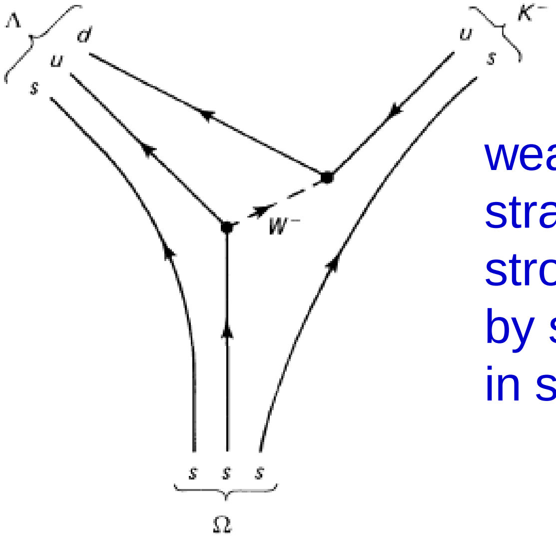
neutron decay



in Δ decays (resonances)
the strong decay channel
dominates over the weak







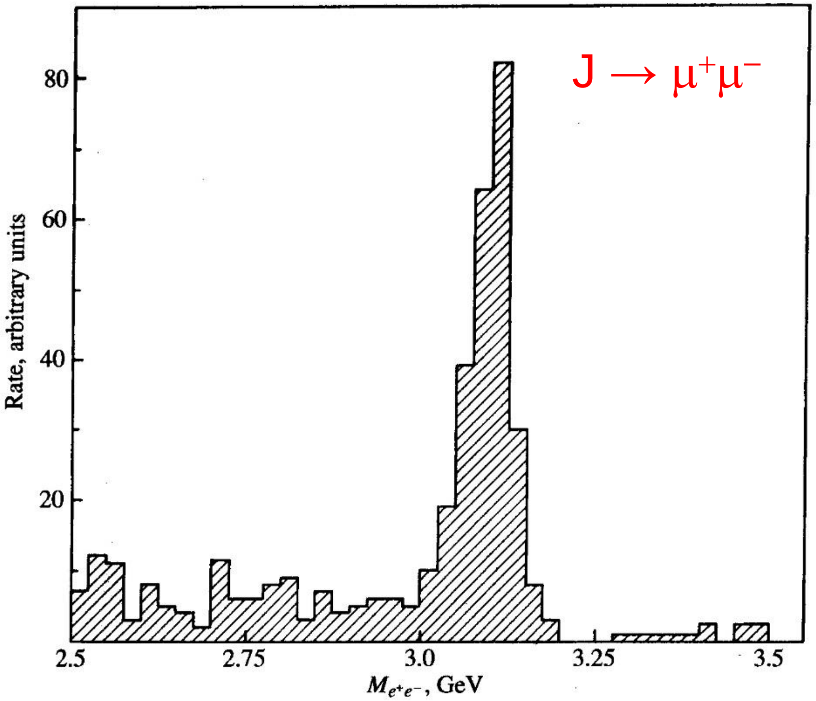
weak decays of
strange baryons:
strong decays are forbidden
by strangness conservation
in strong interactions

Discovery of Charm (J / ψ in 1973)

Charm observed in 1973 as
J / ψ = ($c\bar{c}$) (hidden charm)

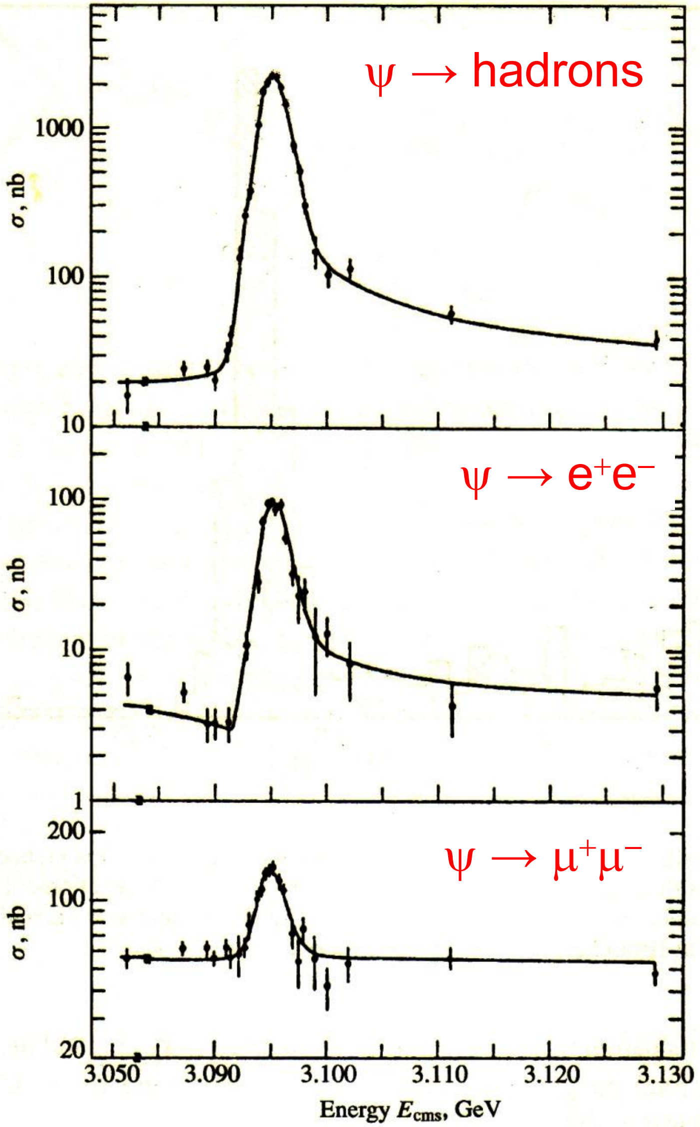
$M_{J/\psi}$ ~ 3096 MeV Γ = 11 keV $J^{PC} = 1^{--}$

1976



$J \rightarrow \mu^+\mu^-$

J : hadroproduction p + Be → J + X



$\psi \rightarrow$ hadrons

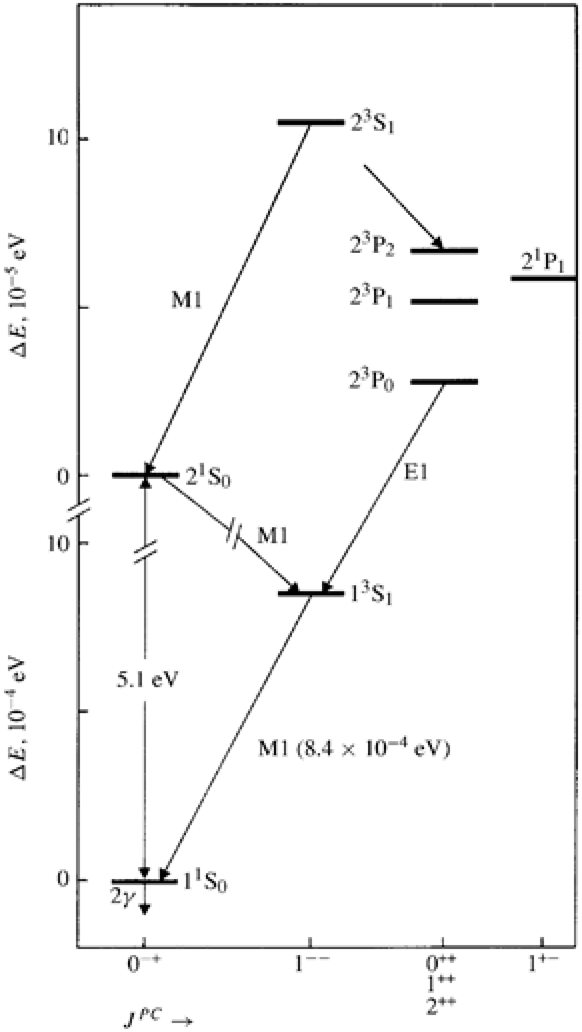
$\psi \rightarrow e^+e^-$

$\psi \rightarrow \mu^+\mu^-$

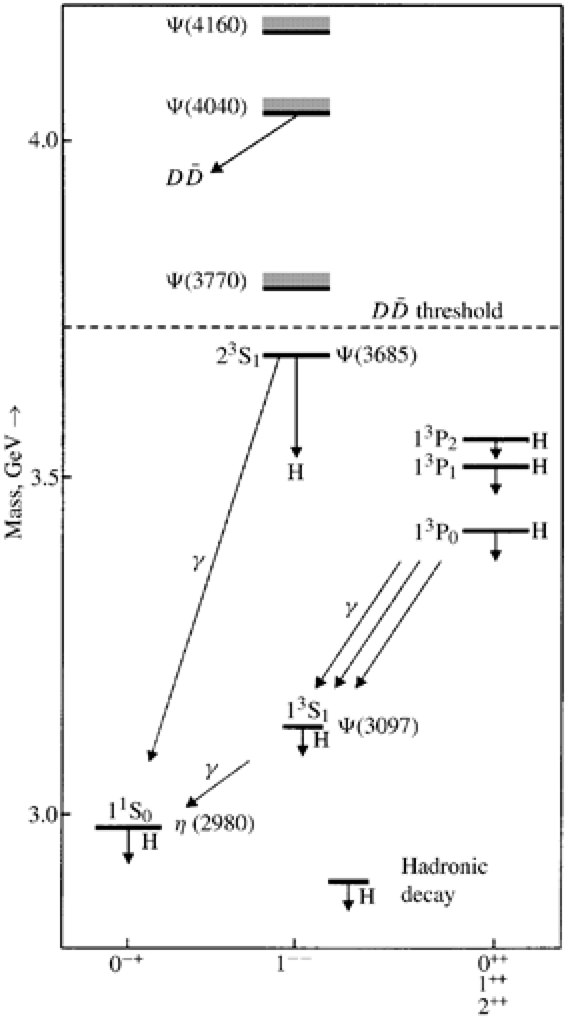
ψ : $e^+e^- \rightarrow \psi \rightarrow$ hadrons
(or e^+e^- or $\mu^+\mu^-$)

Positronium vs $Q\bar{Q}$-onium Levels

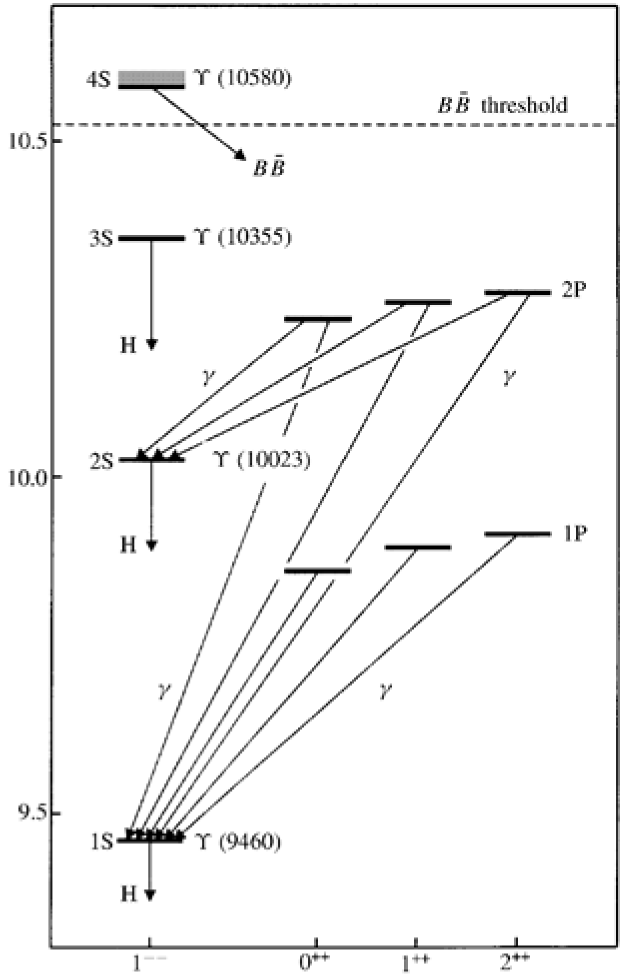
positronium



charmonium Ψ ($c\bar{c}$)



bottonium ϒ ($b\bar{b}$)



Charmonium and Bottonium resonance spectra very similar to the positronium.

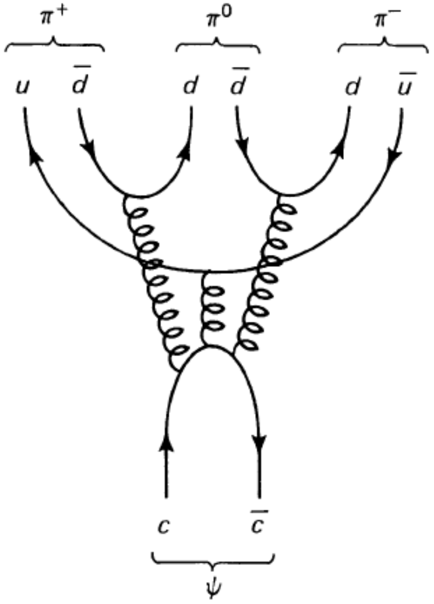
Below the $2m_D$ and $2m_B$ thresholds these states are very narrow.

Okubo-Zweig-Iizuka (OZI) Rule

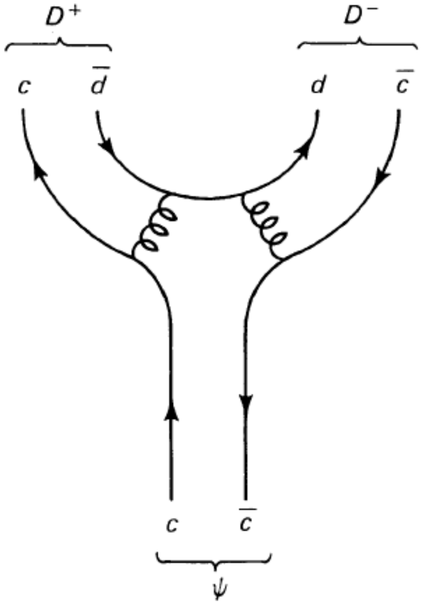
Why is the width of the J / ψ resonance so narrow ?

$M_{J/p}$ = 3097 MeV

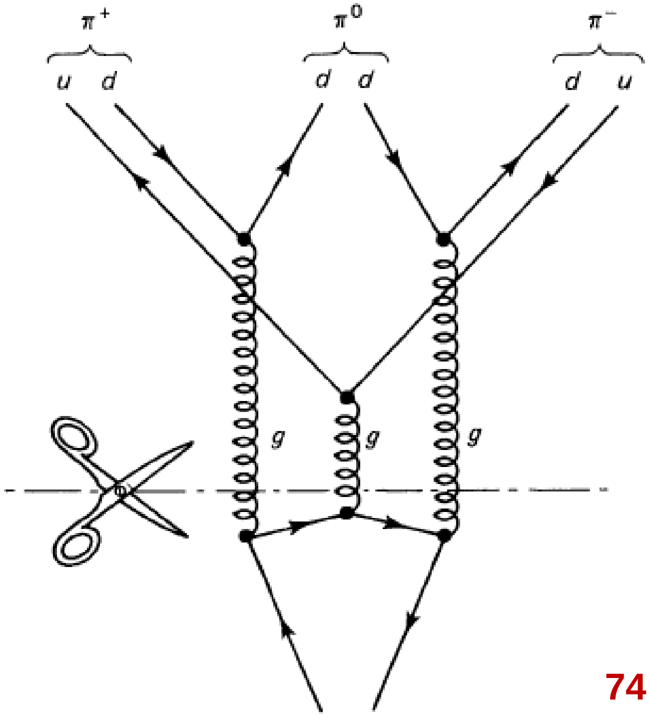
M_D = 1869 MeV



below 2 × m_D threshold
highly suppressed
(narrow resonance)
because of OZI rule



kinematically allowed for
ψ resonances > 2 × m_D
(broad resonances)



OZI rule: if the diagram can be cut in two
by slicing only gluon lines (and not cutting any
external line) the process is supressed

QCD Potential

Charmonium levels are similar to positronium levels →

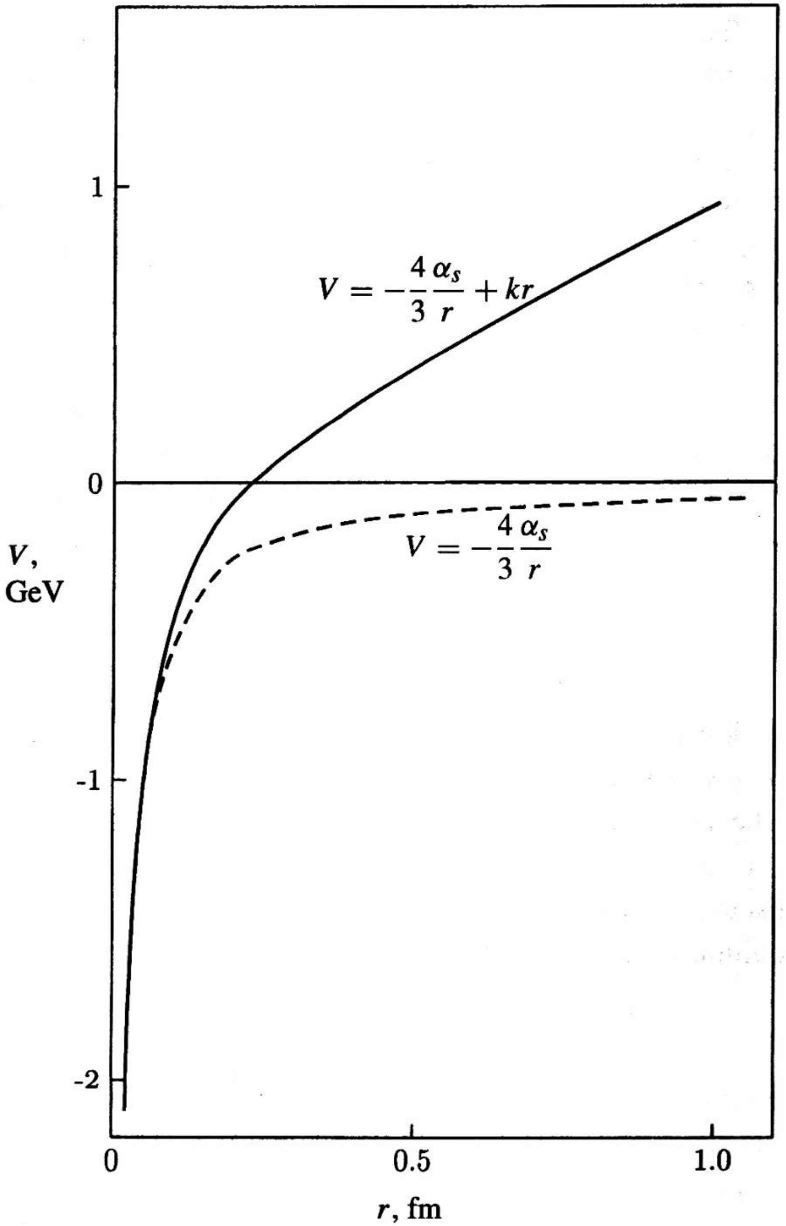
potential of the form 1/r at short distances + confining harmonic potential of the form $F_0 r$ at large distances

empirical QCD potential

$$\boxed{V(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} + F_0 r}$$

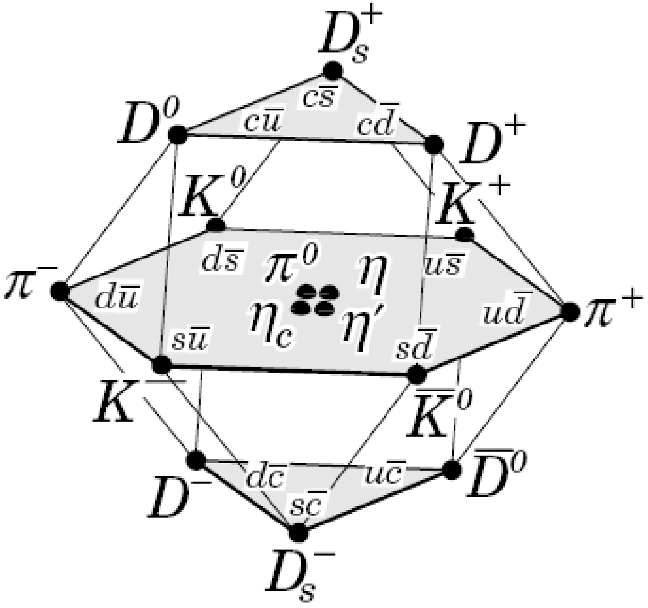
F_0 ~ 900 MeV / fm   (i.e. ~ 16 tons)

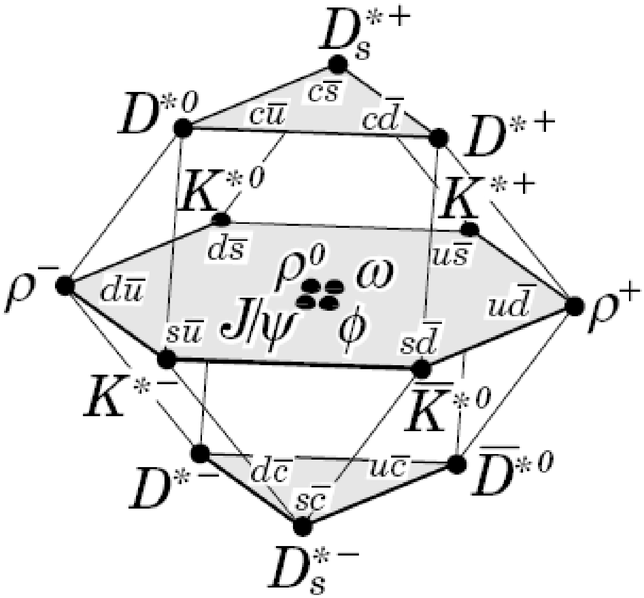
In QCD the strong force at short distances is assumed to have a similar space-time structure to QED.

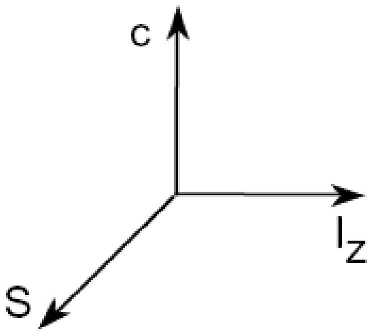


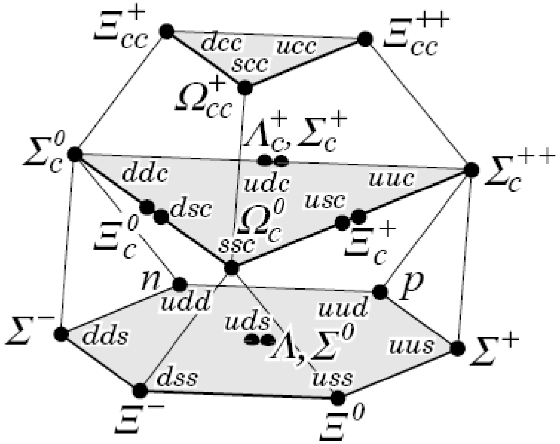
Charm Hadrons

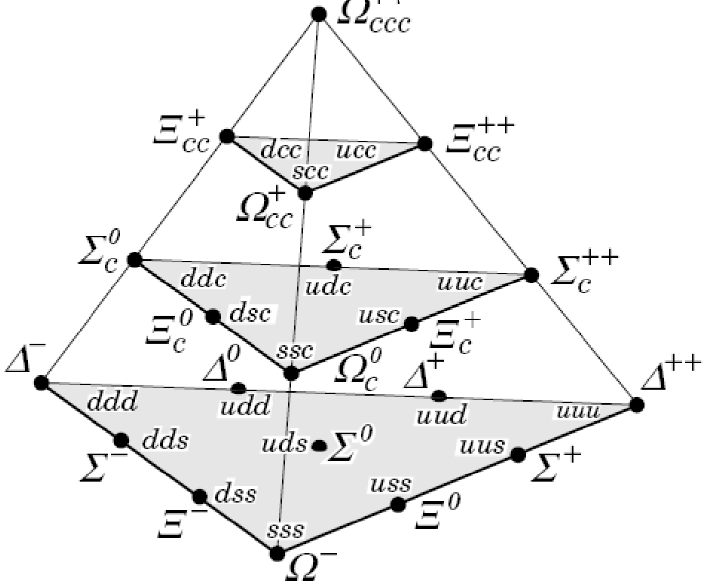
add 4th quark → SU(4)











For Next Week

Study the material and prepare / ask questions
Study ch. 2 in Halzen & Martin and / or ch. 9 in Thomson

Do the homeworks

Next week we will study QCD
  have a first look at the lecture notes, you can already have questions
  read ch. 14 (sec. 3 and 4) in Halzen & Martin and / or ch. 10 in Thomson