Advanced Particle Physics 2

Strong Interactions
and Weak Interactions

L3 – Introduction to QCD

(http://dpnc.unige.ch/~bravar/PPA2/L3)

lecturer

Alessandro Bravar Alessandro.Bravar@unige.ch

tél.: 96210 bureau: EP 206

assistant

Jorge Sabater Jorge.Sabateriglesias@unige.ch

March 7, 2023

Yukawa Theory (1935)

The first theory of Strong Interaction was proposed by Yukawa in 1935: the strong interaction between nucleons (protons and neutrons) is mediated by the exchange of a new particle, the π meson (pion).

interactions between protons, between neutrons, and between protons and neutrons
$\rightarrow$ 3 charges for the mediator: π^+, π^0, π^-

1949

short, finite range $\lambda \sim 1$ fm
$\rightarrow$ potential of the form
(Yukawa potential)

$$\phi(r) \propto \frac{e^{-r/\lambda}\hbar c}{r}$$

The mass of the mediator $m = 1/\lambda$ is inversely proportional to the range of the force $\rightarrow m_\pi \sim 200$ MeV

[recall that in natural units $1 = \hbar \times c = 197.3$ MeV fm (~ 200 MeV fm)]

In 1937 the muon was discovered in cosmic rays and it was wrongly interpreted as the Yukawa π meson. By studying the interactions of this particle with matter (μ lifetime expt.), in 1947 it was shown that this particle cannot be the Yukawa π meson, since it does not interact strongly. In the same year (1947) the pion was positively identified in cosmic rays by observing the decay chain $\pi \rightarrow \mu \rightarrow e$.

Evidence For Color

What about color? Is it only a mathematical expedient to save Pauli exclusion principle?
$\Rightarrow$ associated with strong force mediated by gluons obeying $SU(3)_C$ symmetry (exact)

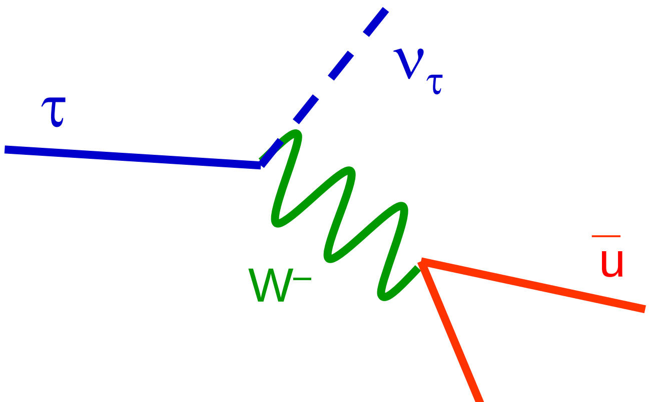
τ decay

$\tau \to \nu_\tau + W^-$

$\to \quad (e\ \nu_e) \quad (\mu\ \nu_\mu) \quad (\bar{u}\ d)$

branching ratio ~ 20% 20% 60%

(3 colors)



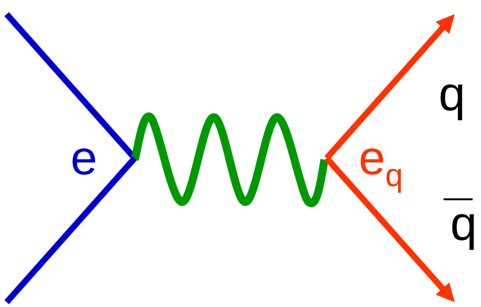
$d' = \cos\theta_C\, d + \sin\theta_C\, s$

π^0 decay

$$\Gamma(\pi^0 \to \gamma + \gamma) = N_c^2 (Q_u^2 - Q_d^2) \frac{\alpha^2 m_\pi^3}{32\pi^2 f_\pi^2}$$

e$^+$e$^-$ annihilation into quarks (hadrons)

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_C \frac{\sum_f Q_f^2 \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} \left(1 + \frac{\alpha_S}{\pi} + \ldots\right)$$

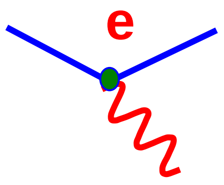


Color in QCD

The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "color" charges

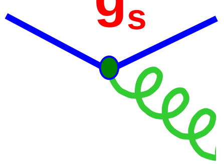
In QED

- the electron carries one unit of electric charge **–e**
- the anti-electron carries one unit of "anti"-charge **+e**
- the force is mediated by a massless "gauge boson" – the photon



In QCD

- quarks carry a color charge **r**, or **g**, or **b**
- anti-quarks carry an anti-color charge $\bar{\mathbf{r}}$, or $\bar{\mathbf{g}}$, or $\bar{\mathbf{b}}$
- the force is mediated by massless "gauge bosons" – the gluons



In QCD, the strong interaction is invariant under rotations in *SU(3)* color space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colors

$\Rightarrow$ *SU(3)* color symmetry

This is an exact symmetry, unlike the approximate uds flavor symmetry.
Don't confuse $SU(3)_c$ color and $SU(3)_F$ flavor, though the Lie group algebra is the same:

- F – flavor, which is not an exact symmetry (it is broken) by the different quark masses
- C – color, exact symmetry $\Rightarrow$ 8 different colored (charged) massless gluons

$SU(3)_c$ Color Symmetry

QCD: each quark flavor exists in 3 colors (red, green, blue or i, j, k) and obey exact *SU(3)* symmetry (they have the same mass!)

fundamental representation	*combinations*		multiplets
$q = \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix}$	$q \otimes \bar{q}$	$(3 \otimes \bar{3})$	$1 \oplus 8$
	$q \otimes q \otimes q$	$(3 \otimes 3 \otimes 3)$	$1 \oplus 8 \oplus 8 \oplus 10$

singlets

all observed hadrons are color singlets (i.e. carry no color)

$$q\bar{q} = \sqrt{\frac{1}{3}}\left(r\bar{r} + g\bar{g} + b\bar{b}\right)$$

$$qqq = \sqrt{\frac{1}{6}}\left(rgb - grb - rbg + gbr + brg - bgr\right)$$

color singlet: $\varepsilon_{abc} q_a q_b q_c$

i.e. they are invariant under transformations (rotations) in *SU(3)* color space

$$q \rightarrow q' = Uq = e^{i\sum_{a=1}^{8} \vartheta_a T^a} q$$

$$q_j \rightarrow q_j{}' = 1 + i\sum_{k=1}^{3}\left(\sum_{a=1}^{8} \vartheta_a T^a\right)_{jk} q_k$$

$T^a = 1/2\, \lambda^a$ are the eight generators of the *SU(3)* transformations, θ_a are 8 parameters.

From QED to QCD

Suppose there is another fundamental symmetry of Nature, say

"invariance under SU(3) local phase transformations"

i.e. require invariance under $\psi \to \psi' = \psi e^{i\sum_{a=1}^{8} \vartheta_a(x) T^a}$

$\theta_a(x)$ are 8 parameters (functions of x) taking different values at each point in space-time.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$ wave function is now a vector in COLOR SPACE $\Rightarrow$ **QCD**

QCD is fully specified by requiring invariance under *SU(3)* local phase transformations. That corresponds to rotating states in color space about an axis whose direction is different at every space-time point

$\Rightarrow$ interaction vertex $-\frac{1}{2} i g_s \lambda^a \gamma^\mu$

g_S is a new coupling constant associated with strong interactions.

Predicts 8 massless gauge bosons – the gluons (one for each generator).

Also predicts exact form for interactions between gluons, i.e. the 3 and 4 gluon vertices.

Gluons

The gluons mediate the QCD force between color charges at different space-time points.

The gluons belong to an *SU(3)* multiplet with color combinations allowed by group theory. Since a $\overline{q}q$ pair can annihilate into a gluon, the gluons must correspond to a color combination equivalent to

$$q \otimes \overline{q} \;:\; \left(3 \otimes \overline{3}\right) \;=\; 1_{\text{singlet}} \oplus 8_{\text{octet}}$$

They form an octet

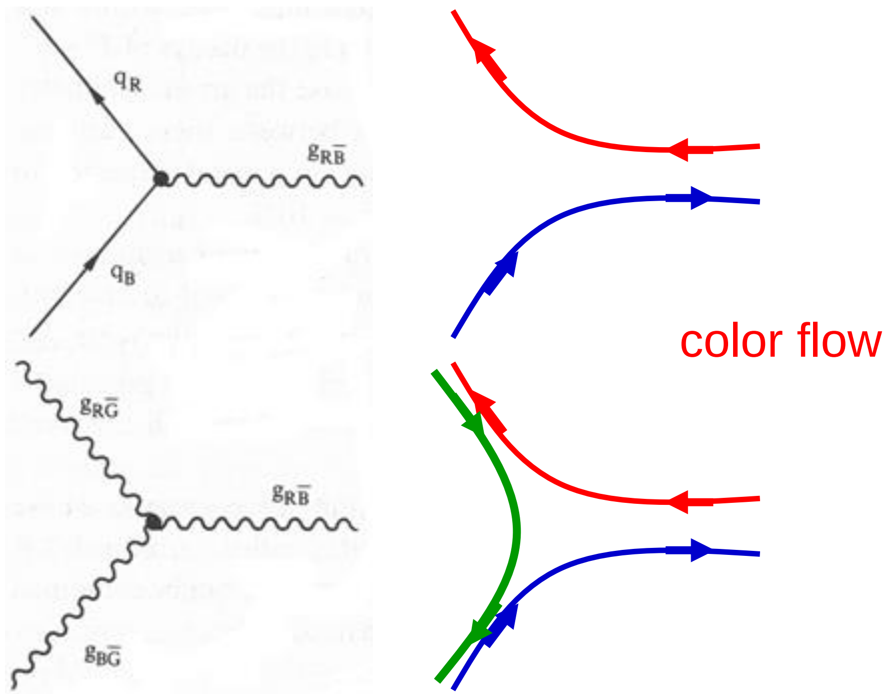
$$R\overline{G},\quad R\overline{B},\quad G\overline{R},\quad G\overline{B},\quad B\overline{R},\quad B\overline{G},\quad \frac{1}{\sqrt{2}}\left(R\overline{R}-G\overline{G}\right),\quad \frac{1}{\sqrt{6}}\left(R\overline{R}+G\overline{G}-2B\overline{B}\right)$$

where each gluon is a color-anticolor combination.

Since they carry color, they can interact among themselves.

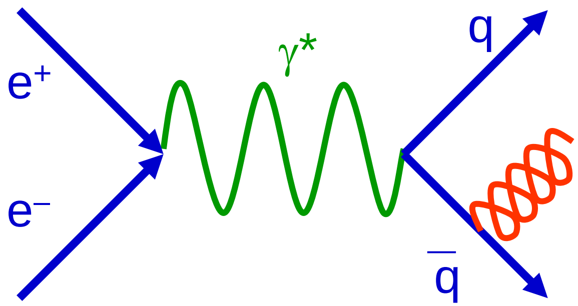
The color singlet $\frac{1}{\sqrt{3}}\left(r\overline{r}+g\overline{g}+b\overline{b}\right)$

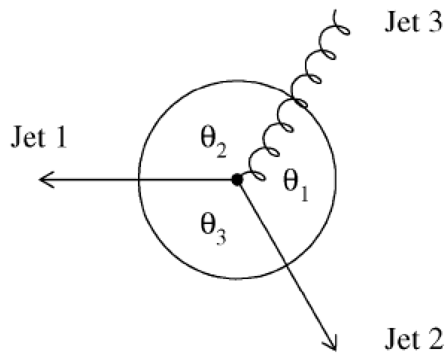
does not carry color and does not play a role (it could give rise to long range effects, since it would not be confined; such effects have not been observed)



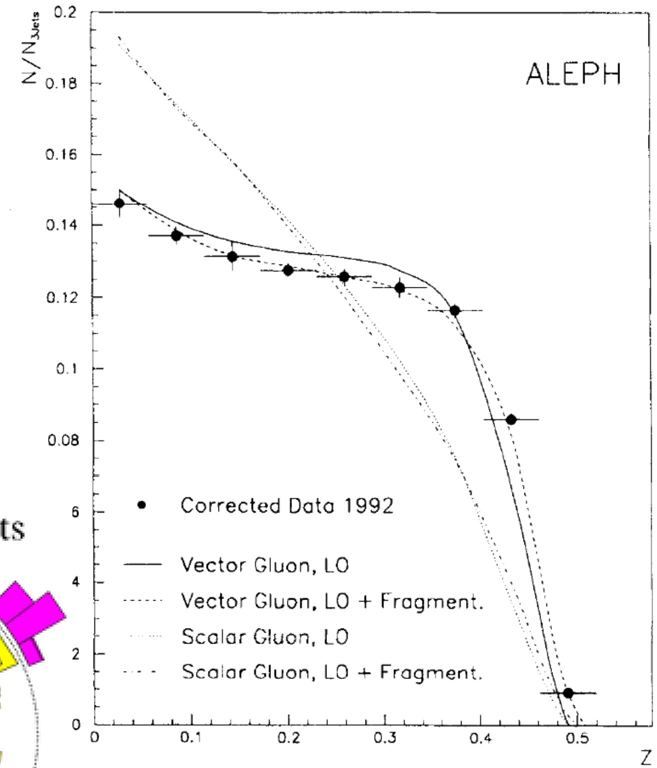
Direct Observation of Gluons

$e^+e^- \to$ 3 jets

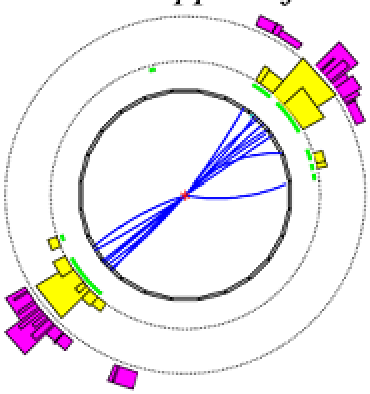




angular distribution of 3rd jet

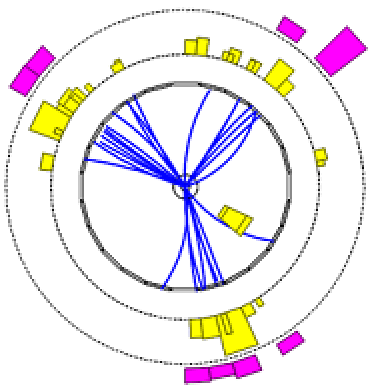


$e^+e^- \to q\bar{q} \to$ 2jets



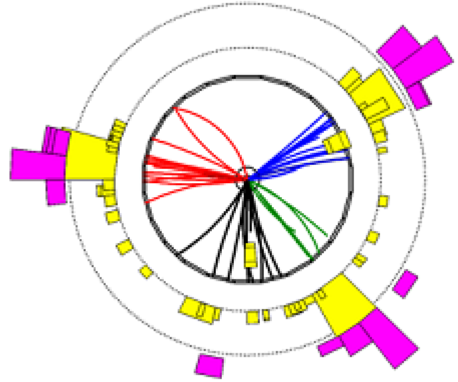
2-jet event

$e^+e^- \to q\bar{q}g \to$ 3jets



3-jet event

$e^+e^- \to q\bar{q}gg \to$ 4jets



4-jet event

data indicate that
gluons have spin 1

The QED Lagrangian

A free electron ψ is described by the Dirac Lagrangian $L_0 = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$

L is invariant under global phase transformations $\psi \rightarrow \psi' = e^{i\alpha}\psi$
(global gauge invariance $\rightarrow$ conservation of charge)
but not under local phase transformations. $\psi \rightarrow \psi' = e^{i\alpha(x)}\psi$

To restore local gauge invariance we introduce the covariant derivative

$$D_\mu \equiv \partial_\mu + ieA_\mu(x)$$

which must transform as ψ:

$$D_\mu\psi \rightarrow \left(D_\mu\psi\right)' = D_\mu{}'\psi' = \left(\partial_\mu + ieA_\mu{}'\right)e^{i\alpha(x)}\psi = e^{i\alpha(x)}\left(\partial_\mu + i\partial_\mu\alpha\left(x\right) + ieA_\mu{}'\right)\psi = e^{i\alpha(x)}D_\mu\psi$$

To form the covariant derivative we introduce a vector field A_μ (the photon field) that transforms as

$$A_\mu(x) \rightarrow A_\mu(x)' = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$$

e measures the coupling strength of the Dirac field to this vector field (***e*** is the elementary charge ***e*=¦*e*¦**, electron charge ***-e***) . And the Lagrangian becomes

$$L = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - m\bar{\psi}\psi$$

where $L_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu$ is the interaction term between the Dirac field and the A_μ field

To complete the Lagrangian we have still to add a term corresponding to the kinetic energy of the photon field

$$L_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Since this term must be gauge invariant as well, it can involve only the gauge invariant filed strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Note that a mass term $m^2 A_\mu A^\mu$ for the photon field is not gauge invariant, and thus the photon must be massless [$m_\gamma < 3 \times 10^{-27}$ eV from galactic magnetic fields].

Adding all pieces, leads to the QED Lagrangian

$$L_{QED} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Note that the Lagrangian

$$L = -e\bar{\psi}\gamma^\mu \psi A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

leads to the Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$

Local gauge invariance does not lead to new conservation laws,
it states that charges are conserved locally.
That has deep implications $\Rightarrow$ it fixes the form of the interaction!

- All the interactions between fermions and spin-1 bosons in the Standard Model are specified by the principle of local gauge invariance.

- Changing phases arbitrarily at different space-time points will create phase differences which might become observable, if not compensated by some mechanism.

This is the role of the new A_μ vector field that we have introduced to restore the gauge invariance of the Lagrangian (theory).

- By demanding local phase invariance to preserve the invariance of the Lagrangian we are forced to introduce a vector filed A_μ, the gauge field, which couples to the Dirac particle in exactly the same way as the photon field (Maxwell eq.).

- A mass term $\frac{1}{2} m^2 A_\mu A^\mu$ is forbidden by gauge invariance, therefore the photon must be massless, and the gauge field will have infinite range.

The fermion mass term $m\bar{\psi}\psi$ does not brake local gauge invariance because left-handed and right-handed fields transform in the same way (not the case for weak interactions).

- By imposing the “natural” requirement of local gauge invariance (local phase invariance) on the free Lagrangian, we are led to the interacting field theory of QED.

Historically, things evolved the other way around: starting from Maxwell equations we developed QED, then discovered local phase invariance.

- Gauge invariance is a very powerful symmetry that fixes the form of the interaction between the fundamental matter particles (fermions) and the field quanta. It has become one of the most basic and essential ingredients of modern particle physics.

Any new theory describing particle interactions is required to satisfy this requirement.

- The QCD symmetry group SU(3) is sufficiently similar to U(1) – QED, so that the same principles can be applied to QCD, and sufficiently different to describe rich new physics

The QCD Lagrangian

Within the SM, the strong interaction is described by QCD, which is a local gauge theory built on the non-Abelian internal symmetry of quarks known as color – that is *$SU(3)_C$*. In QCD the strong force at short distances is assumed to have a similar space-time structure to QED.

To derive the QCD Lagrangian we proceed in analogy to QED.

Let $$q(x) = \begin{pmatrix} q_R(x) \\ q_B(x) \\ q_G(x) \end{pmatrix}$$ be the quark fields, where each of the q_i is a Dirac spinor with color *i*.

The unitary transformations which mix quarks of different colors are generated by the elements of the SU(3) Lie algebra, λ_a, a = 1, … , 8 = color index and corresponds to one of the 8 Gell-Mann matrices.

The most general transformation amongst the colors is induced by the unitary operator

$$U = \exp\left[i\theta_a T^a\right]$$ with $T_a = \lambda_a/2$ the generators of the transformation and

$\theta_1, \ldots \theta_8$ are eight real parameters associated with the transformation. The quark field *q*(x) then transforms as

$$q(x) \to q'(x) = e^{i\vartheta_a T^a} q(x)$$

Local gauge invariance requires that the Lagrangian is invariant under an arbitrary phase transformation at each point in space-time – i.e. $\theta_a = \theta_a(x)$.

Note that observables such as electric charge are invariant under *SU(3)*$_C$ transformations

$$Q = e\sum_f d^3x \frac{1}{\sqrt{3}}\left(\frac{2}{3}u_i^+u_i - \frac{1}{3}d_i^+d_i - \frac{1}{3}s_i^+s_i + \ldots\right)$$

The free quark Lagrangian is given by

$$L = i\bar{q}\gamma^\mu\partial_\mu q - m\bar{q}q \longrightarrow \text{Dirac equation}$$

Invariance under global color transformation leads to 8 conserved charges and currents

$$q \to q' = \left(1 + i\sum_{a=1}^{8}\vartheta_a\frac{\lambda_a}{2}\right)q \longrightarrow j_a^\mu(x) = \bar{q}(x)\frac{\lambda_a}{2}\gamma^\mu q(x)$$

This Lagrangian however is not invariant under local phase transformations of the quarks fields in the *SU(3)* color space

$$q(x) \to q'(x) = e^{i\sum_a \vartheta_a(x)\frac{\lambda_a}{2}}q(x)$$

Local gauge invariance of *L* can be restored by introducing the covariant derivative D_μ

$$\partial_\mu \to D_\mu \equiv \partial_\mu + ig_S\sum_a\frac{\lambda_a}{2}G_\mu^a \qquad \partial_\mu q \to D_\mu q \equiv \left(\partial_\mu + ig_S\sum_a\frac{\lambda_a}{2}G_\mu^a\right)q$$

involving eight new Lorentz-vector fields, i.e. 8 massless vector fields $G_a^\mu(x)$ (the gluons) with g_S the new strong coupling constant $\alpha_S = g_S^2 / 4\pi$.

This leads to the following Lagrangian

$$L = \bar{q}(i\gamma^{\mu}\partial_{\mu} - m)q - g_s(\bar{q}\gamma^{\mu}\frac{\lambda_a}{2}q)G_{\mu}^{a}$$

Let's examine an infinitesimal transformation of the quark current $\bar{q}\gamma^{\mu}\frac{\lambda_a}{2}q$

$$\bar{q}\gamma^{\mu}\frac{\lambda_a}{2}q \to \bar{q}'\gamma^{\mu}\frac{\lambda_a}{2}q' = U^{\dagger}\bar{q}\gamma^{\mu}\frac{\lambda_a}{2}Uq = (1 - i\vartheta_b\frac{\lambda_b}{2})\bar{q}\gamma^{\mu}\frac{\lambda_a}{2}(1 + i\vartheta_b\frac{\lambda_b}{2})q$$

Ignoring higher order terms, we obtain

$$\bar{q}\gamma^{\mu}\frac{\lambda_a}{2}q + i\vartheta^{b}\bar{q}\gamma^{\mu}(\frac{\lambda_a}{2}\frac{\lambda_b}{2} - \frac{\lambda_b}{2}\frac{\lambda_a}{2})q = \bar{q}\gamma^{\mu}\frac{\lambda_a}{2}q - f_{abc}\vartheta_b\bar{q}\gamma^{\mu}\frac{\lambda_c}{2}q$$

$$\left(\lambda_a\lambda_b - \lambda_b, \lambda_a\right) = \left[\lambda_a, \lambda_b\right] = i2f_{abc}\lambda_c$$

which turns out to be not invariant because the generators do not commute.

The additional term is compensated for by G_{μ}^{a} .

This has an impact on how the gluon field G_{μ}^{a} should transform.

The basic physics requirement is that the covariant derivative transforms as ψ (as in QED)

$$D_{\mu}\psi \to \left[D_{\mu}\psi\right]' = D_{\mu}'\psi' = \mathrm{e}^{i\sum_a \vartheta_a(x)\lambda^a/2}\left(D_{\mu}\psi\right)$$

Assume that G_{μ}^{a} transforms as

$$G_{\mu}'^{a} = G_{\mu}^{a} + \delta G_{\mu}^{a}$$

Develop separately LHS and RHS (ignoring higher order terms)

$$D'_\mu \psi' \approx \left(\partial_\mu + i g_s \frac{\lambda_a}{2} G'^a_\mu \right)\left(1 + i\vartheta^b \frac{\lambda_b}{2} \right)\psi$$

$$= \left(\partial_\mu + i g_s \frac{\lambda_a}{2} G^a_\mu + i g_s \frac{\lambda_a}{2} \delta G^a_\mu \right)\left(1 + i\vartheta^b \frac{\lambda_a}{2} \right)\psi$$

$$= \left(\partial_\mu + i \partial_\mu \vartheta^b \frac{\lambda_b}{2} + i g_s \frac{\lambda_a}{2} G^a_\mu - g_s \frac{\lambda_a}{2} G^a_\mu \vartheta^b \frac{\lambda_b}{2} + i g_s \frac{\lambda_a}{2} \delta G^a_\mu \right)\psi$$

$$e^{i\vartheta_a \lambda^a/2}\left(D_\mu \psi \right) \approx \left(1 + i\vartheta_a \frac{\lambda^a}{2} \right)\left(\partial_\mu + i g_s \frac{\lambda_b}{2} G^b_\mu \right)\psi$$

$$= \left(\partial_\mu + i g_s \frac{\lambda_b}{2} G^b_\mu + i\vartheta_a \frac{\lambda^a}{2} \partial_\mu - \vartheta_a \frac{\lambda^a}{2} g_s \frac{\lambda_b}{2} G^b_\mu \right)\psi$$

and compare LHS and RHS (drop ψ, since it holds for any state)

$$i\partial_\mu \vartheta^a \frac{\lambda_a}{2} - g_s \frac{\lambda_a}{2} G^a_\mu \vartheta^b \frac{\lambda_b}{2} + i g_s \frac{\lambda_a}{2} \delta G^a_\mu = i\vartheta^a \frac{\lambda_a}{2} \partial_\mu - g_s \vartheta^a \frac{\lambda_a}{2} \frac{\lambda_b}{2} G^b_\mu$$

Develop separately LHS and RHS (ignoring higher order terms)

$$D'_\mu \psi' \approx \left(\partial_\mu + i g_s \lambda_a G'^a_\mu / 2\right)\left(1 + i\vartheta^b \lambda_b / 2\right)\psi$$

$$= \left(\partial_\mu + i g_s \lambda_a G^a_\mu / 2 + i g_s \lambda_a \delta G^a_\mu / 2\right)\left(1 + i\vartheta^b \lambda_b / 2\right)\psi$$

$$= \left(\partial_\mu + i\partial_\mu \vartheta^b \lambda_b / 2 + i g_s \lambda_a G^a_\mu / 2 - g_s \lambda_a G^a_\mu / 2 \vartheta^b \lambda_b / 2 + i g_s \lambda_a \delta G^a_\mu / 2\right)\psi$$

$$e^{i\vartheta_a \lambda^a /2}\left(D_\mu \psi\right) \approx \left(1 + i\vartheta_a \frac{\lambda^a}{2}\right)\left(\partial_\mu + i g_s \frac{\lambda_b}{2} G^b_\mu\right)\psi$$

$$= \left(\partial_\mu + i g_s \frac{\lambda_b}{2} G^b_\mu + i\vartheta_a \frac{\lambda^a}{2}\partial_\mu - \vartheta_a \frac{\lambda^a}{2} g_s \frac{\lambda_b}{2} G^b_\mu\right)\psi$$

and compare LHS and RHS

$$i\partial_\mu \vartheta^a \frac{\lambda_a}{2} - g_s \frac{\lambda_a}{2} G^a_\mu \vartheta^b \frac{\lambda_b}{2} + i g_s \frac{\lambda_a}{2}\delta G^a_\mu = i\vartheta^a \frac{\lambda_a}{2}\partial_\mu - g_s \vartheta^a \frac{\lambda_a}{2}\frac{\lambda_b}{2} G^b_\mu$$

We can then solve for δG

$$\lambda_a \delta G^a_\mu = -\frac{1}{g_s}\partial_\mu \vartheta^a \lambda_a - i\lambda_a \vartheta^a \frac{\lambda_b}{2} G^b_\mu + i\frac{\lambda_b}{2} G^b_\mu \vartheta^a \lambda_a$$

$2if_{abc}\lambda_c$

$$= -\frac{1}{g_s}\partial_\mu \vartheta^a \lambda_a - \frac{i}{2}\vartheta^a G^b_\mu \left[\lambda_a \lambda_b - \lambda_b \lambda_a\right] = -\frac{1}{g_s}\partial_\mu \vartheta^a \lambda_a + f_{abc}\vartheta^a G^b_\mu \lambda_c$$

We can then solve for δG

$$\lambda_a \delta G_\mu^a = -\frac{1}{g_s}\partial_\mu \vartheta^a \lambda_a - i\lambda_a \vartheta^a \frac{\lambda_b}{2} G_\mu^b + i\frac{\lambda_b}{2} G_\mu^b \vartheta^a \lambda_a$$

$$= -\frac{1}{g_s}\partial_\mu \vartheta^a \lambda_a - \frac{i}{2}\vartheta^a G_\mu^b \left[\lambda_a \lambda_b - \lambda_b \lambda_a\right] = -\frac{1}{g_s}\partial_\mu \vartheta^a \lambda_a + f_{abc}\vartheta^a G_\mu^b \lambda_c$$

$2if_{abc}\lambda_c$

And finally

$$\delta G_\mu^a = -\frac{1}{g_s}\partial_\mu \vartheta^a + f_{abc}\vartheta^b G_\mu^c$$

So we conclude that we can restore the gauge invariance of the Lagrangian, provided that the gluon field transforms as

$$G_\mu^a(x) \rightarrow G_\mu^{a\prime} = G_\mu^a(x) - \frac{1}{g_S}\partial_\mu \vartheta^a(x) + f_{abc}\vartheta^b(x) G_\mu^c(x)$$

The appearance of the term $f_{abc}\theta_b G_c^\mu$ is due to the non-Abelian nature of the *SU(3)* gauge group (the generators of the group do not commute).

It is believed that from this follows the most remarkable properties of the strong interactions: the asymptotic freedom and confinement.

Note that there is no obvious historical guide like for QED (Maxwell Eq.).

In analogy to QED we can introduce the gluon field tensor $\partial_\mu G_\nu^a - \partial_\nu G_\mu^a$

However this term is not gauge invariant, because the gluon fields do not commute.
A mass term for the gluon field would also violate the gauge invariance.
Instead, if we define the gluon field tensor as

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f_{abc} G_\mu^b G_\nu^c$$

the gluon “kinetic” term

$$L = -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}$$

is gauge invariant. Also in this case we had to add an additional term, which as we will see, describes the self interactions of gluons.

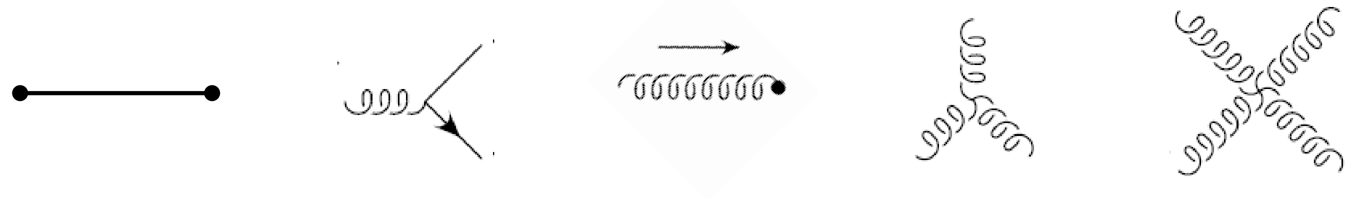
The QCD Lagrangian density finally becomes

$$L_{QCD} = \bar{q}\left(i\gamma^\mu \partial_\mu - m\right)q - g_S\left(\bar{q}\gamma^\mu \frac{\lambda_a}{2} q\right) G_\mu^a - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}$$

This Lagrangian density is the natural consequence of local color symmetry.
It looks like QED, but the gluon field energy tensor implies completely new features:
the gluons self interact via a tri-linear coupling proportional to g_S and
a four-linear coupling proportional to g_S^2.

Symbolically, we can rewrite the QCD Lagrangian as

$$L_{QCD} = \text{"}q\bar{q}\text{"} + g_S\text{"}q\bar{q}G\text{"} + \text{"}G^2\text{"} + g_S\text{"}G^3\text{"} + g_S^2\text{"}G^4\text{"}$$



The first three terms have QED analogues. They describe

i) the free propagation of quarks,

ii) the quark-gluon interaction, and

iii) the free propagation of gluons.

The last two terms are “new” and indicate the presence of

i) three gluon vertices and

ii) four gluon vertices

and reflect the fact that gluons themselves carry color charge and can interact among themselves.

The gauge invariance determines uniquely the structure of these gluon self-coupling terms and forbids higher multi-gluon couplings.
Note that the same coupling constant g_S couples the gluon fields to themselves and the gluons to quark fields.

Quark – Gluon Interaction (Vertex)

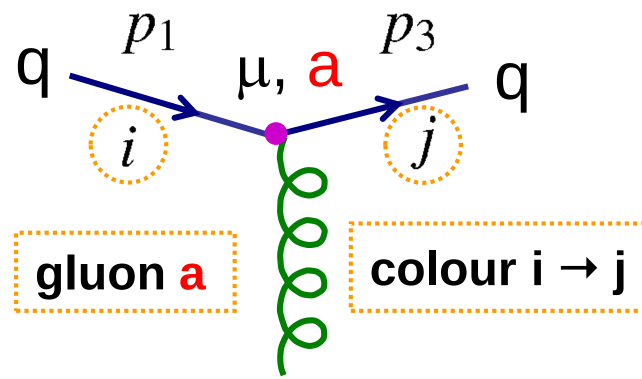
Let's represent the colour part of the quark wave functions by

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad b = c3_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and quark wave functions by $u_i(p) \rightarrow c_i u(p)$

The QCD *qqg* vertex can be written as

$$\boxed{\bar{u}(p_3) c_j^\dagger \left\{ -i g_s \frac{\lambda^a}{2} \gamma^\mu \right\} c_i u(p_1)}$$



The only "difference" w.r.t. QED is the insertion of the *3* × *3 SU(3)* Gell-Mann matrices

Then let's develop the color part
(this is just one entry in the λ matrix)

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ji}$$

Finally, the fundamental quark-gluon interaction can be written

$$\boxed{\bar{u}(p_3) c_j^\dagger \left\{ -i g_s \frac{\lambda^a}{2} \gamma^\mu \right\} c_i u(p_1) \equiv \bar{u}(p_3) \left\{ -i g_s \frac{\lambda^a_{ji}}{2} \gamma^\mu \right\} u(p_1)}$$

Quarks interact with the gluon by exchanging color charges at the interaction vertex.

Gluon Field Tensor

As we saw, the gauge invariant gluon field energy tensor is given by

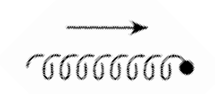
$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f_{abc} G^b_\mu G^c_\nu$$

Let's develop the kinetic term

$$L = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

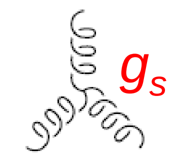
gluon "propagator"

$$L_G^{(2)} = \left(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu\right)\left(\partial^\mu G^\nu_a - \partial^\nu G^\mu_a\right)$$



triple gluon coupling

$$L_G^{(3)} = -\frac{g_S}{2} f_{abc}\left(\delta_\mu G^a_\nu - \delta_\nu G^a_\mu\right) G^\mu_b G^\nu_c$$



quartic gluon coupling

$$L_G^{(4)} = -\frac{g_S^2}{2} f_{abe} f_{cde} G_{a\mu} G_{b\nu} G^\mu_c G^\nu_d$$

Gluon – Gluon Interaction

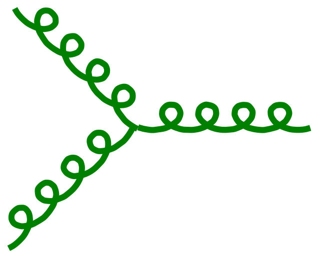
In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral)

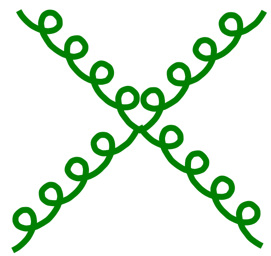
In QCD the gluons do carry color charge

⇨ **Gluon Self-Interactions**

Two new vertices (no QED analogues)

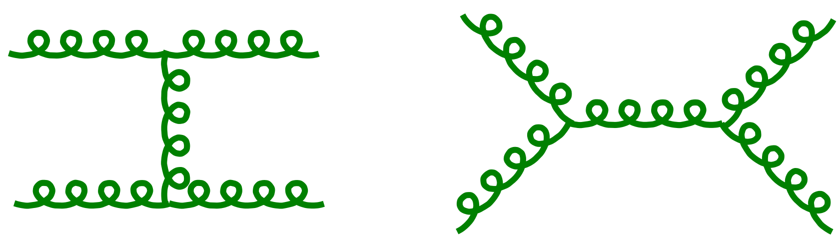
triple gluon vertex



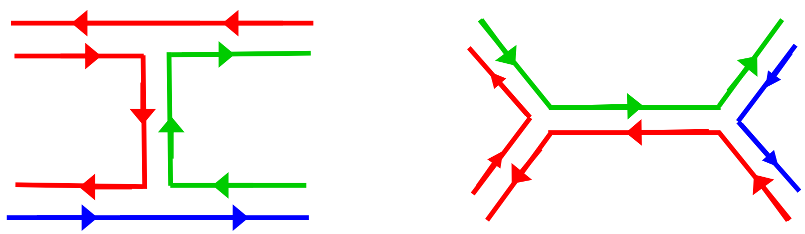


quartic gluon vertex

In addition to quark-quark scattering, we can also observe gluon-gluon scattering:



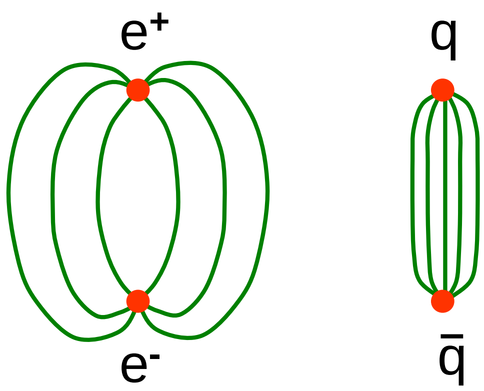
e.g. possible way of arranging the color flow



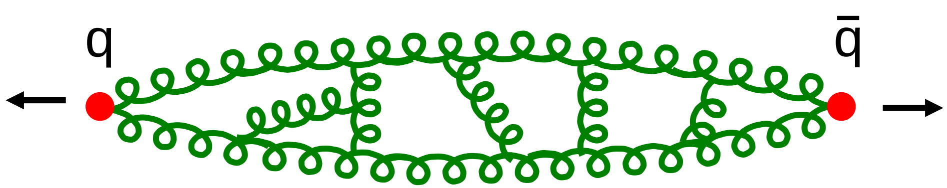
Gluon Self-Interaction and Confinement

Gluon self-interactions are believed to give rise to color confinement.
Qualitative picture:

- compare QED with QCD
- in QCD gluon self-interactions squeeze lines of force into a "flux tube"



What happens when we try to separate two colored objects e.g. a $q\bar{q}$ pair?



A flux tube of interacting gluons of approximately constant energy density ~1 GeV/fm is formed

$$\Rightarrow \quad V(r) \sim F_0 r$$

Require infinite energy to separate colored objects to infinity, i.e. to free the quarks.
Colored quarks and gluons are always confined within colorless states.
In this way QCD provides a plausible explanation of confinement –
but not yet proven (although there has been recent progress with Lattice QCD).

Not everything is that simple though !

The QCD Lagrangian thus obtained is not suitable for quantization. The derivation of Feynman rules from L_{QCD} and their use is non-trivial because of complications in handling quantization and gauge invariance compared to QED.

We have to introduce a gauge fixing condition (similar in QED). This term is necessary for the existence of a (free) gluon propagator equivalent to a covariant gauge $\partial^{\mu} G^{a}_{\mu\nu} = 0$. The gluon fields can be expressed in a variety of gauges; graphs involving gluon loops (in particular helicity 0 contributions) introduce unphysical polarization degrees of freedom in observables. To suppress these unphysical states *ghosts* have been introduced. In axial gauges (more complicated gluon propagator) ghosts do not appear.

In principle, L_{QCD} could contain a further term, which is gauge invariant:

$$L_{\vartheta} = \vartheta_{\mathrm{QCD}} \frac{g_s^2}{32\pi} G_a^{\mu\nu} \tilde{G}^a_{\mu\nu} \qquad \tilde{G}^a_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\rho} G^a_{\lambda\rho}$$

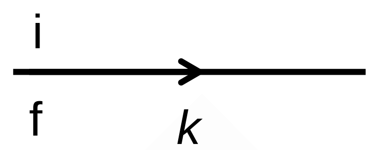
where θ_{QCD} is the QCD θ parameter, which violates P, T, and CP (neutron dipole mom.)
$\Rightarrow$ Strong CP (violation) problem
so far no evidence for this term: $\theta_{QCD} < 10^{-11}$
θ_{QCD} is one of the 19 parameters of the Standard Model!

For massless quarks there is no scale in the QCD Lagrangian. Left handed quarks decouples from right handed quarks. This would lead to a duplication of all hadron states which has not been observed (spontaneous chiral symmetry breaking).

Feynman Rules for QCD

external lines

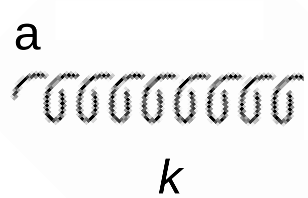
quarks



in: $u_f^i\,(k)$

out: $\bar{u}_f^i\,(k)$

gluons

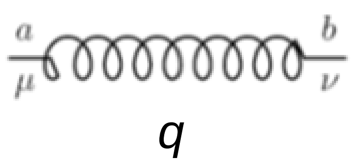


in: $\varepsilon_a^{\mu}\,(k)$

out: $\varepsilon_a^{\mu\,*}\,(k)$

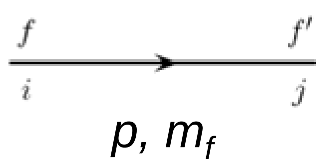
propagators

gluon
(Feynman gauge)



$$\frac{-ig^{\mu\nu}\delta^{ab}}{q^2}$$

quark



$$\frac{i\delta^i_j\delta^f_{f'}}{\not{p}-m_f}$$

μ, ν Lorentz indices

a, b, c gluon color "indices"

i, j quark color indices

f, f´ flavor indices

g – coupling constant
$\alpha_S = g^2/4\pi$

Feynman Rules for QCD - Vertices

μ, ν Lorentz indices

a, b, c gluon color “indices”

i,j quark color indices

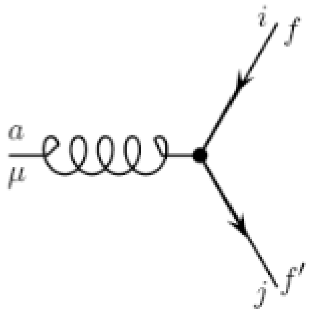
f, f´ flavor indices

g_s – coupling constant
$\alpha_S = g^2/4\pi$

λ_a generators *SU(3)* group

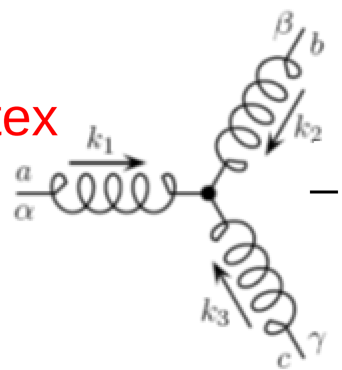
f^{abc} anti-symmetric SU(3) group structure constants

quark – gluon vertex



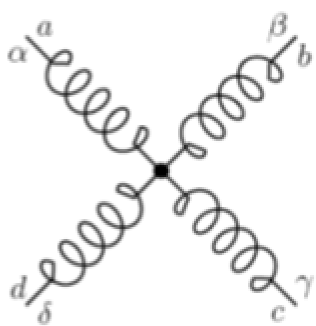
$$-ig_s\gamma^{\mu}\delta_f^{f'}\frac{\lambda^a_{ji}}{2}$$

triple – gluon vertex



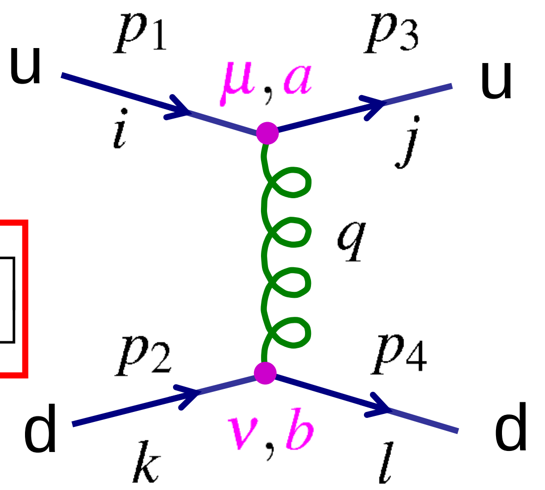
$$-g_s f_{abc}\left[g^{\alpha\beta}\left(k_1-k_2\right)^{\gamma}+g^{\beta\gamma}\left(k_2-k_3\right)^{\alpha}+g^{\gamma\alpha}\left(k_3-k_1\right)^{\beta}\right]$$

quartic – gluon vertex



$$-ig_s^2\left[\begin{array}{l} f_{abc}f_{cde}\left(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}\right)\\ +f_{ace}f_{bde}\left(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\gamma\beta}\right)\\ +f_{ade}f_{bce}\left(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\gamma}g^{\delta\beta}\right)\end{array}\right]$$

QCD vs QED



QED

$$-iM = \left[\bar{u}(p_3)\left\{-ie\gamma^{\mu}\right\}u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}(p_4)\left\{-ie\gamma^{\nu}\right\}u(p_2)\right]$$

QCD

$$-iM = \sum_a \left[\bar{u}(p_3)\left\{-c_j^{\dagger} i g_s \frac{\lambda^a}{2}\gamma^{\mu} c_i\right\}u(p_1)\right]\frac{-ig_{\mu\nu}\delta_{ab}}{q^2}\left[\bar{u}(p_4)\left\{-c_l^{\dagger} i g_s \frac{\lambda^b}{2}\gamma^{\nu} c_k\right\}u(p_2)\right]$$

$$= \sum_a \left[\bar{u}(p_3)\left\{-ig_s \frac{\lambda^a_{ji}}{2}\gamma^{\mu}\right\}u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}(p_4)\left\{-ig_s \frac{\lambda^a_{lk}}{2}\gamma^{\nu}\right\}u(p_2)\right]$$

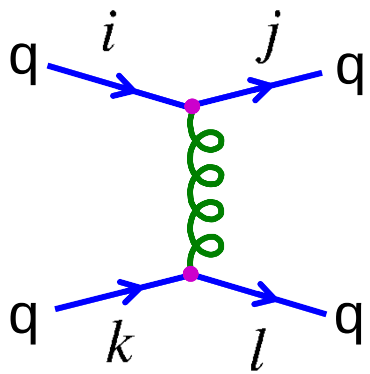
QCD Matrix Element = QED Matrix Element with

$e^2 \rightarrow g_s^2$ or equivalently $\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}$

QCD Matrix Element includes an additional "colour factor"

$$C_F(ik \rightarrow jl) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

Color Factors



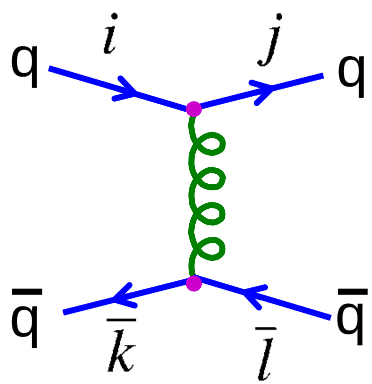
$$C_F\left(ik \to jl\right) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda_{ji}^{a}\lambda_{lk}^{a}$$

note $C_F\left(rg \to br\right) = 0$

$$C_F\left(rr \to rr\right) = \frac{1}{3}$$

$$C_F\left(rg \to rg\right) = -\frac{1}{6}$$

$$C_F\left(rg \to gr\right) = \frac{1}{2}$$

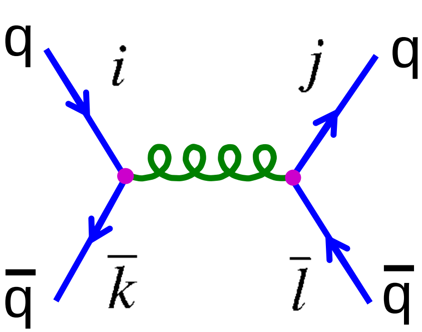


$$C_F\left(i\bar{k} \to j\bar{l}\right) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda_{ji}^{a}\lambda_{kl}^{a}$$

$$C_F\left(r\bar{r} \to r\bar{r}\right) = \frac{1}{3}$$

$$C_F\left(r\bar{g} \to r\bar{g}\right) = -\frac{1}{6}$$

$$C_F\left(r\bar{r} \to g\bar{g}\right) = \frac{1}{2}$$



$$C_F\left(i\bar{k} \to j\bar{l}\right) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda_{ij}^{a}\lambda_{kl}^{a}$$

$$C_F\left(r\bar{r} \to r\bar{r}\right) = \frac{1}{3}$$

$$C_F\left(r\bar{r} \to g\bar{g}\right) = -\frac{1}{6}$$

$$C_F\left(r\bar{g} \to r\bar{g}\right) = \frac{1}{2}$$

Since the color is not observable,
sum over final colors and average over initial colors

$$\left\langle |M_t|^2 \right\rangle = \frac{1}{3\times 3} \sum_{colors} \dots$$

recall $\sum_{a=1}^{8} \lambda_{ji}^{a} \lambda_{lk}^{a} = 2\delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}$

calculate

$$\sum_{colors} \dots = \sum_{a,b,i,j,k,l} \left(\frac{\lambda_{ij}^{a}}{2} \frac{\lambda_{kl}^{a}}{2} \right)^{+} \left(\frac{\lambda_{ij}^{b}}{2} \frac{\lambda_{kl}^{b}}{2} \right) = \sum_{a,b} \sum_{i,j} \left(\frac{\lambda_{ij}^{a}}{2} \frac{\lambda_{ji}^{b}}{2} \right) \cdot \sum_{k,l} \left(\frac{\lambda_{kl}^{a}}{2} \frac{\lambda_{lk}^{b}}{2} \right) =$$

$$= \sum_{a,b} Tr\left(\frac{\lambda^{a}}{2} \frac{\lambda^{b}}{2} \right) Tr\left(\frac{\lambda^{a}}{2} \frac{\lambda^{b}}{2} \right) = 2$$

it depends on the process
for $\gamma q \rightarrow gq$, $\Sigma_{colors} = 4/3$

i.e.

QED

$$\left\langle |M_t|^2 \right\rangle = \frac{2}{3\times 3} \cdot \frac{8}{2\times 2} \cdot \frac{s^2 + u^2}{t^2}$$

color spin dynamics

QCD Potential

Let's assume that at short distances the potential describing the interaction of a $q\bar{q}$ pair is the same as in electrodynamics, except for the color factor f_C (see quarkonium spectroscopy)

$$V_{q\bar{q}}(r) = -f_C \frac{\alpha_s \hbar c}{r} \qquad f_C = \frac{1}{4}\left(c_3^\dagger \lambda^a c_1\right)\left(c_2^\dagger \lambda^a c_4\right)$$

According to QCD, the $q\bar{q}$ pair is in a color singlet state $\frac{1}{\sqrt{3}}\left(r\bar{r} + g\bar{g} + b\bar{b}\right)$

and the corresponding color factor is $f_C = \frac{1}{4}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\lambda_{ij}^a \lambda_{ji}^a = \frac{4}{3}$

For an octet color configuration, i.e. $\frac{1}{\sqrt{2}}\left(r\bar{r} - b\bar{b}\right)$ one finds $f_C = -\frac{1}{6}$

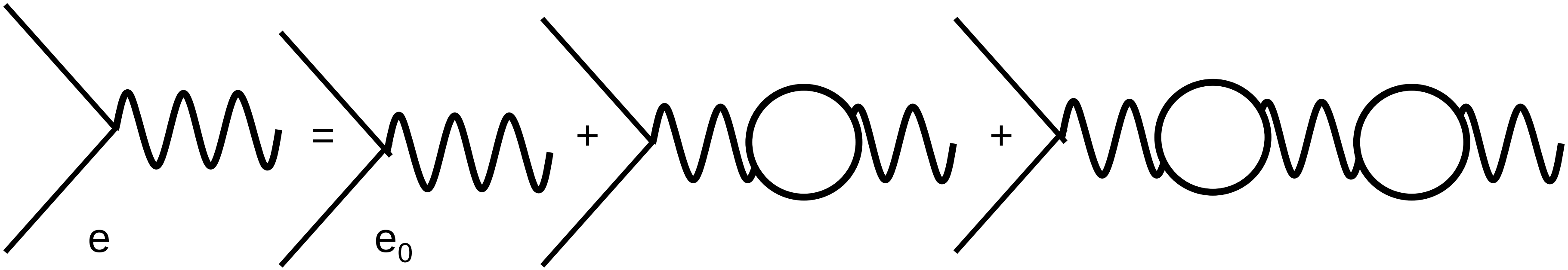
Finally

$$V_{q\bar{q}}(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} \quad \text{color singlet}$$

$$V_{q\bar{q}}(r) = +\frac{1}{6}\frac{\alpha_s \hbar c}{r} \quad \text{color octet}$$

Quarks attract one another most strongly when they are in the color singlet configuration

Renormalization in QED



physical or effective charge | bare charge | bare charge screened by e^+e^- loops

$$\alpha(Q^2) = \alpha_0 \left\{ 1 + \frac{\alpha_0}{3\pi}\log\frac{Q^2}{M^2} + \frac{1}{2}\left(\frac{\alpha_0}{3\pi}\log\frac{Q^2}{M^2}\right)^2 + \ldots \right\} = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\log\frac{Q^2}{M^2}}$$

large Q^2 leading log sum (M cutoff on loop momentum)

subtract

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha_0} - \frac{1}{3\pi}\log\frac{Q^2}{M^2} + \ldots$$

$$\frac{1}{\alpha(\mu^2)} = \frac{1}{\alpha_0} - \frac{1}{3\pi}\log\frac{\mu^2}{M^2} + \ldots$$

$$\frac{1}{\alpha(Q^2)} - \frac{1}{\alpha(\mu^2)} = -\frac{1}{3\pi}\left(\log\frac{Q^2}{M^2} - \log\frac{\mu^2}{M^2}\right) = -\frac{1}{3\pi}\log\frac{Q^2}{\mu^2}$$

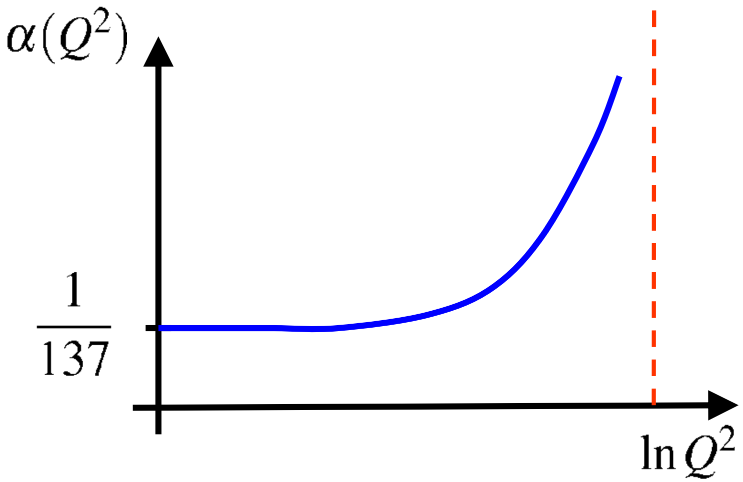
running coupling constant

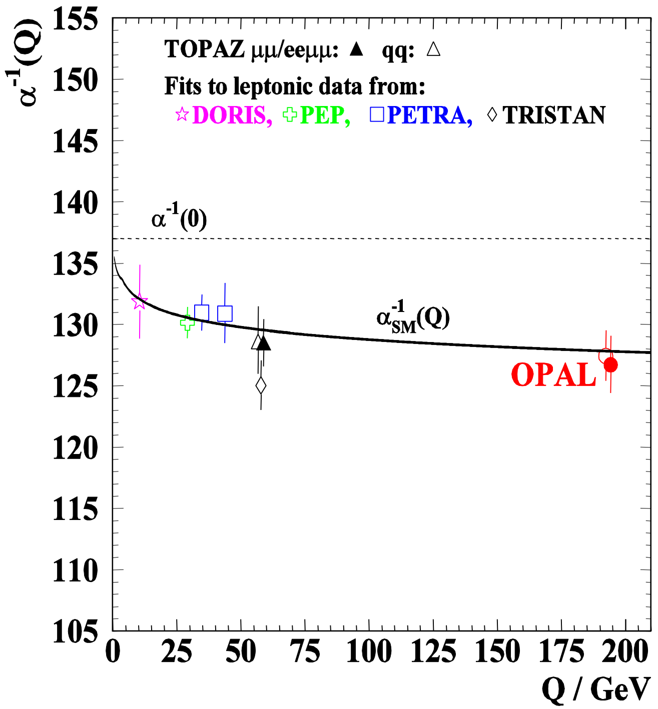
$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log\frac{Q^2}{\mu^2}}$$

infinities removed at the price of introducing the renormalization scale μ

$\alpha(\mu^2 \to 0)$ = measured = 1/137

Running of α_{EM}





Might worry that coupling becomes infinite at

$$\log\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at $Q^2 \sim 10^{52}$ GeV^2

But quantum gravity effects would come in way below this energy scale and it is highly unlikely that QED "as it is" would be valid in this regime.

In QED, running coupling increases very slowly

atomic physics ($Q^2 \sim 0$):

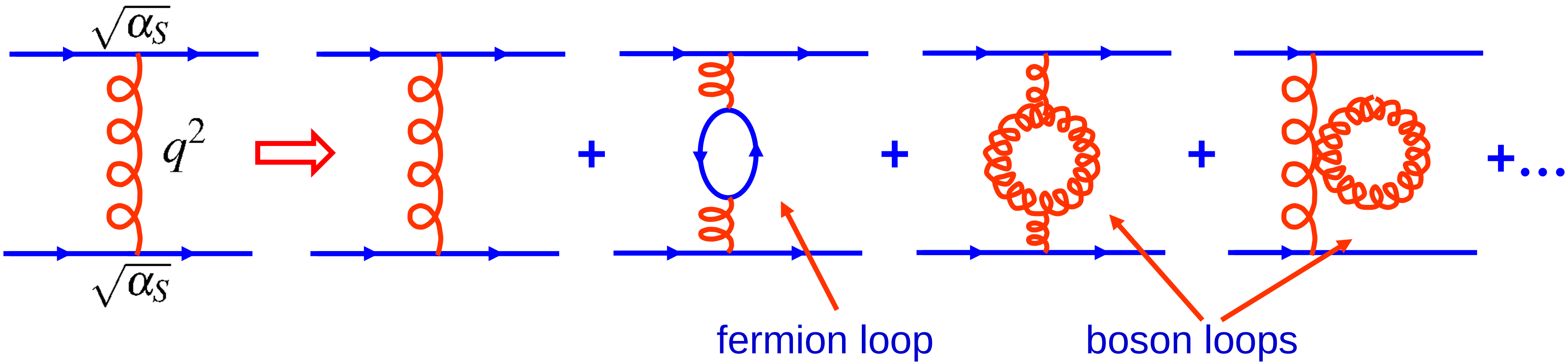
$1/\alpha = 137.03599976(50)$

high energy physics:

$1/\alpha\ (M_Z) = 128$

Renormalization in QCD

Higher order corrections not only from $q\bar{q}$ loops (like in QED) but also from gluon loops.



Running of strong coupling constant

$$\alpha_S\left(Q^2\right)=\alpha_S\left(Q_0^2\right)\Big/\left[1+b_0\alpha_S\left(Q_0^2\right)\log\frac{Q^2}{Q_0^2}\right]$$

in QED $b_0 = -1/3\pi < 0$

$$b_0=-\frac{2n_f}{12\pi}\qquad+\qquad+\frac{11N_c}{12\pi}=\frac{33-2n_f}{12\pi}>0$$

(as long as $n_f < 17$)

the term $-2n_f$ comes from quark loops and behaves as in QED
the term $11N_c$ comes from gluon loops and has a + sign $\Rightarrow b_0 > 0$

Renormalization in QCD

$$\alpha_S(Q^2) = \alpha_S(\mu^2) \Big/ \left[1 + b_0 \alpha_S(Q_0^2) \log \frac{Q^2}{\mu^2}\right]$$

2004

$$Q^2 \frac{\partial \alpha_S}{\partial Q^2} = \beta(\alpha_S) = -b_0 \alpha_S^2$$

known as the QCD beta function

with $$b_0 = \frac{11N_c - 2n_f}{12\pi}$$

N_c = number of colours

N_f = number of quark flavours

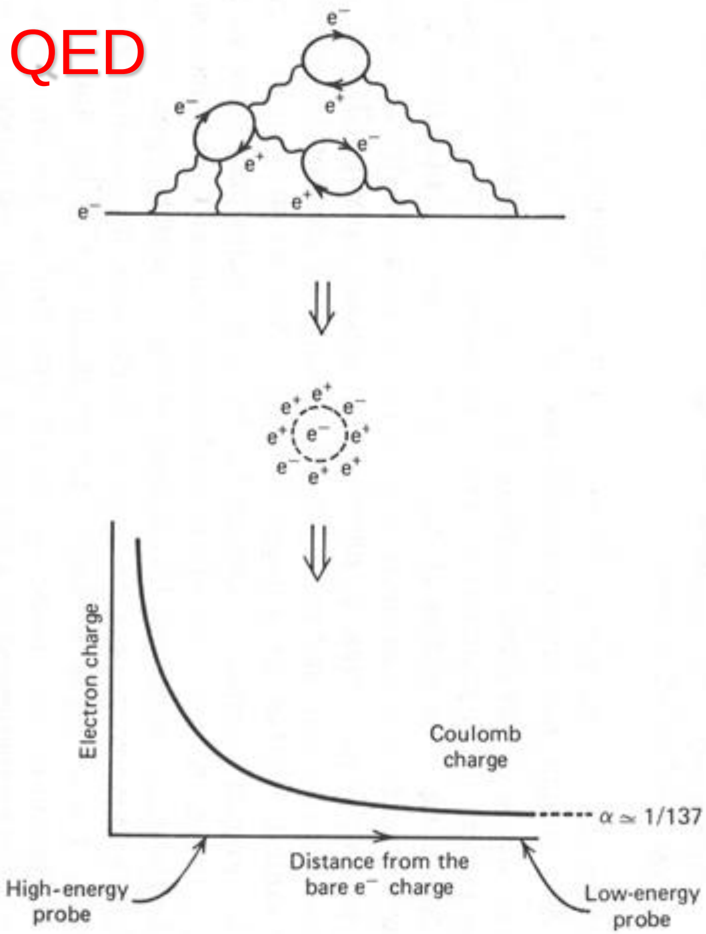
$N_c = 3$ & $N_f = 6$ $\Rightarrow$ $b_0 > 0$ $\Rightarrow$ α_S decreases with increasing Q^2

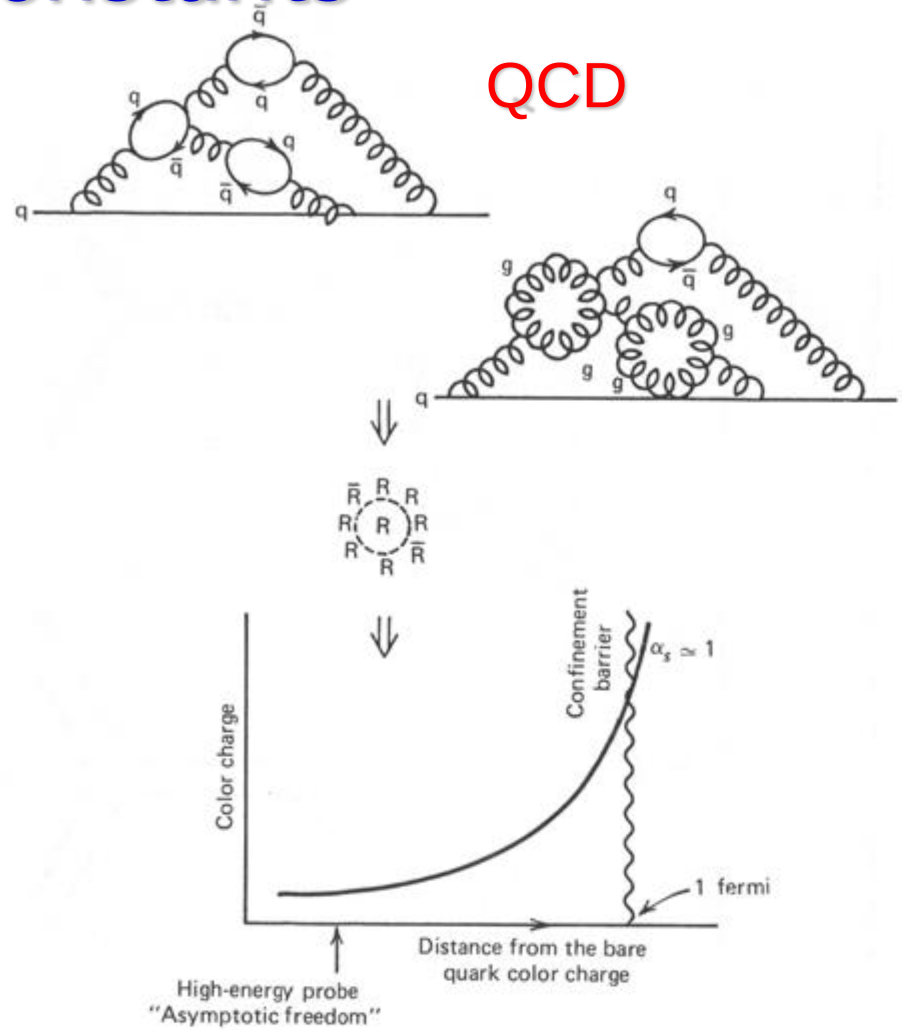
asymptotic freedom

[there is a complication whenever crossing a flavor threshold,
i.e. $N_f = 3 \rightarrow N_f = 4$ at ~1.5 GeV, etc.]

Gluons have an anti-screening effect:
1) gluons can irradiate their charge over space and their charge is not localized
2) the initial quark charge is diffused over space

Running Coupling Constants

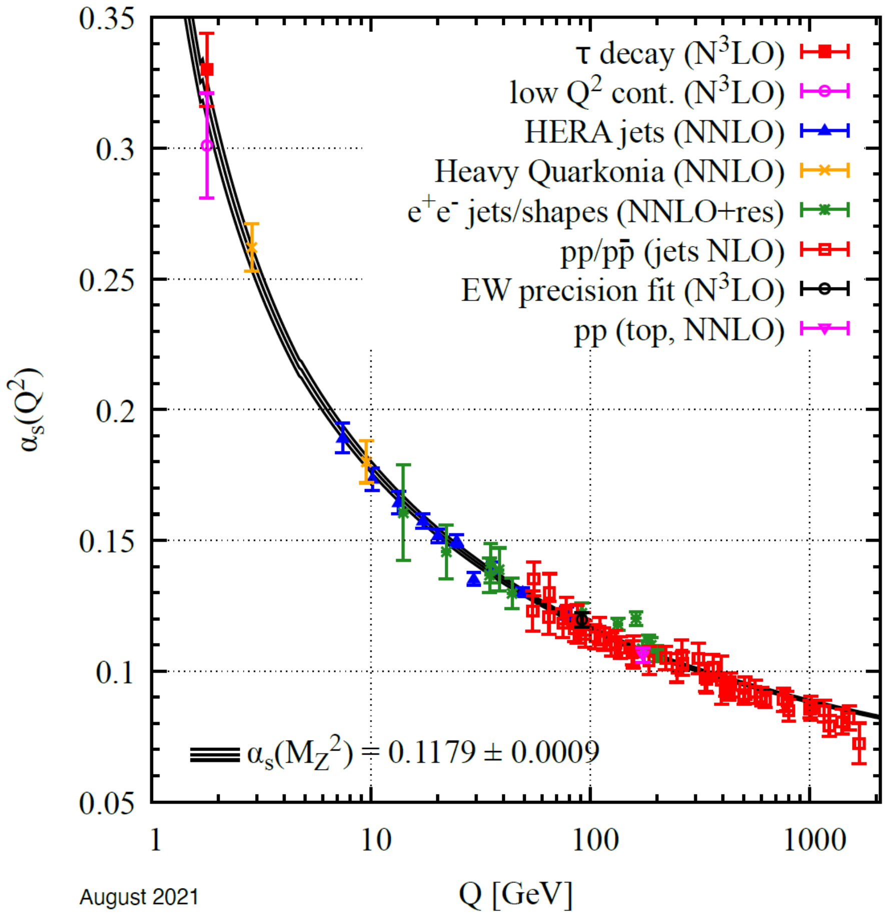




$$\alpha_{EM}\left(Q^2\right)=\frac{\alpha_{EM}\left(\mu^2\right)}{1-\frac{1}{3\pi}\alpha_{EM}\left(\mu^2\right)\ln\frac{Q^2}{\mu^2}}$$

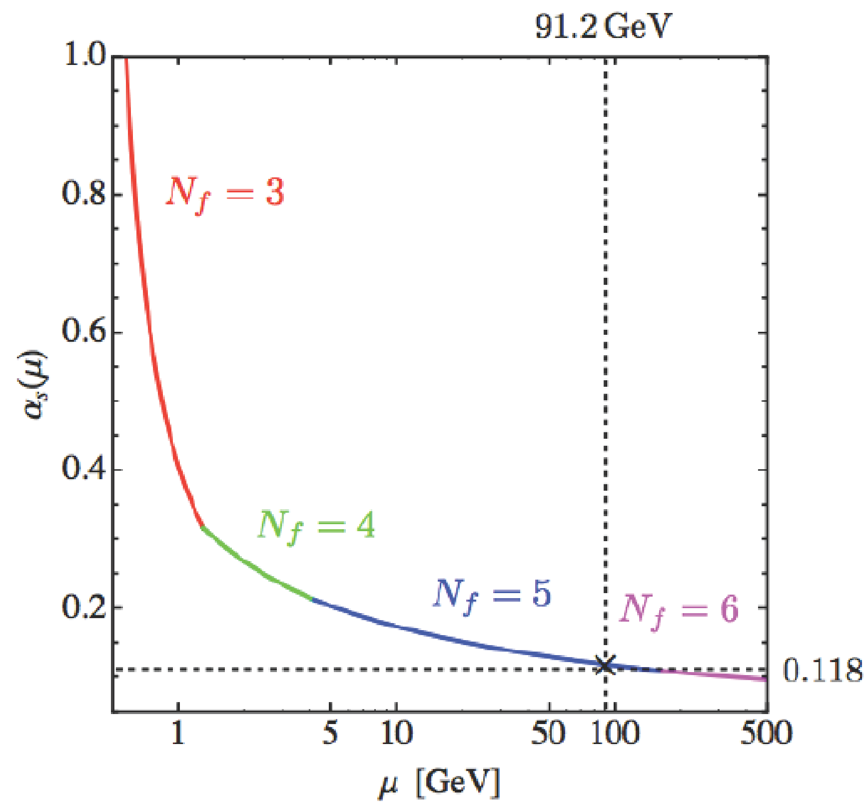
$$\alpha_S\left(Q^2\right)=\frac{\alpha_S\left(\mu^2\right)}{1+\frac{33-2N_f}{12\pi}\alpha_S\left(\mu^2\right)\ln\frac{Q^2}{\mu^2}}$$

Running of α_S



$\alpha_S(\mu)$ at the μ of measurement

active flavors and running of α_S



world average:

$$\alpha_S\left(M_Z^2\right)^{\overline{MS}} = 0.1179 \pm 0.0009$$

Comparing α_S Measurements

Summarizing:

1. can measure α_S in many different processes

2. overall consistent picture:
α_S from very different measurements are compatible

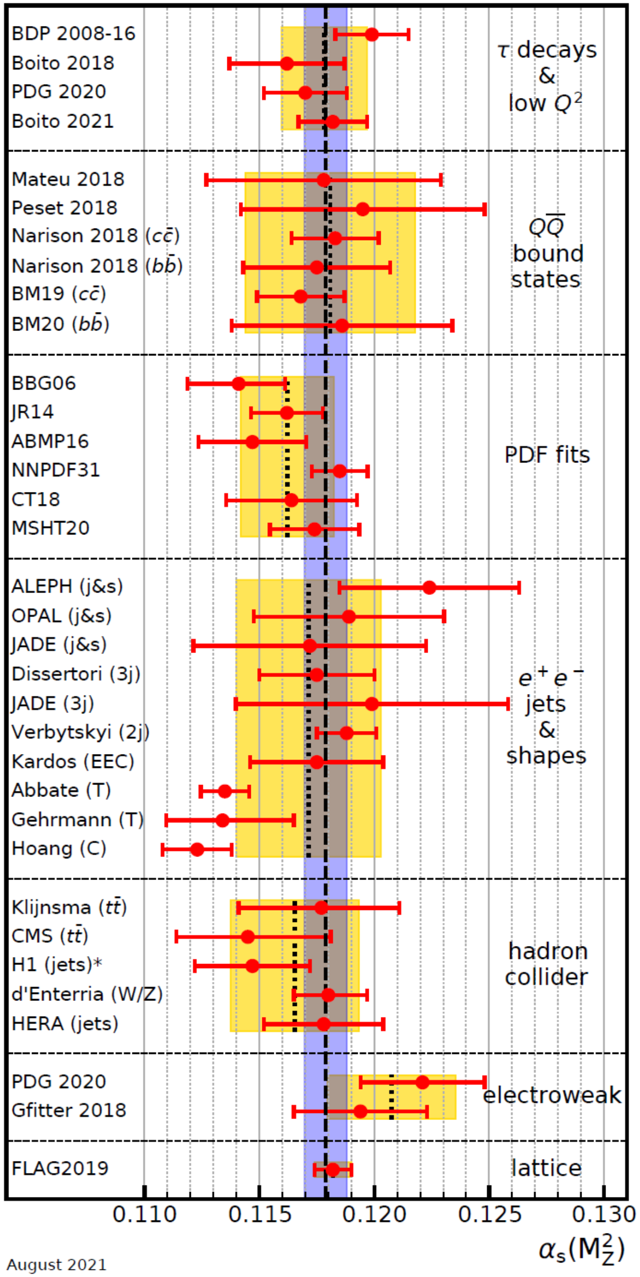
3. α_S decreases slowly with Q^2 (high energy) (logarithmic only)

4. α_S is not that small at current experimental scales (α_S ~ 0.1 – 0.3)

5. higher order corrections are and will remain important

world average

$$\alpha_S(M_Z)^{\overline{MS}} = 0.1179 \pm 0.0009$$



Λ_{QCD}

The QCD Lagrangian makes no mention of the renormalization scale μ (μ is an arbitrary parameter), even though a choice of μ is required to define the theory at the quantum scale and physical observables, like *R*, cannot depend on μ

$$\mu^2 \frac{d}{d\mu^2} R\left(Q^2/\mu^2, \alpha_s\right) = \left(\mu^2 \frac{d}{d\mu^2} + \mu^2 \frac{d\alpha_s}{d\mu^2}\frac{\partial}{\partial\alpha_s}\right) R = 0$$

All the scale dependence in *R* enters through the running of $\alpha_s(Q^2)$.

An alternative approach is to introduce a dimensionful parameter in the definition of $\alpha_s(Q^2)$:

$$\log\frac{Q^2}{\Lambda^2_{QCD}} = -\int_{\alpha_s(Q^2)}^{\infty} \frac{dx}{\beta(x)} \quad \rightarrow \quad \log\Lambda^2_{QCD} = \log\mu^2 - \frac{1}{b_0\alpha_s(\mu^2)}$$

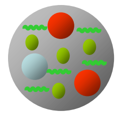
Λ_{QCD} represents the scale at which the coupling would diverge and the perturbative expansion breaks down. This could be an indication that the confinement of quarks and gluons inside hadrons is a consequence of the growing of $\alpha_s(Q^2)$ at small scales. Unfortunately it is hard to determine Λ_{QCD} , Λ_{QCD} ~ 200 MeV (= 1 fm).

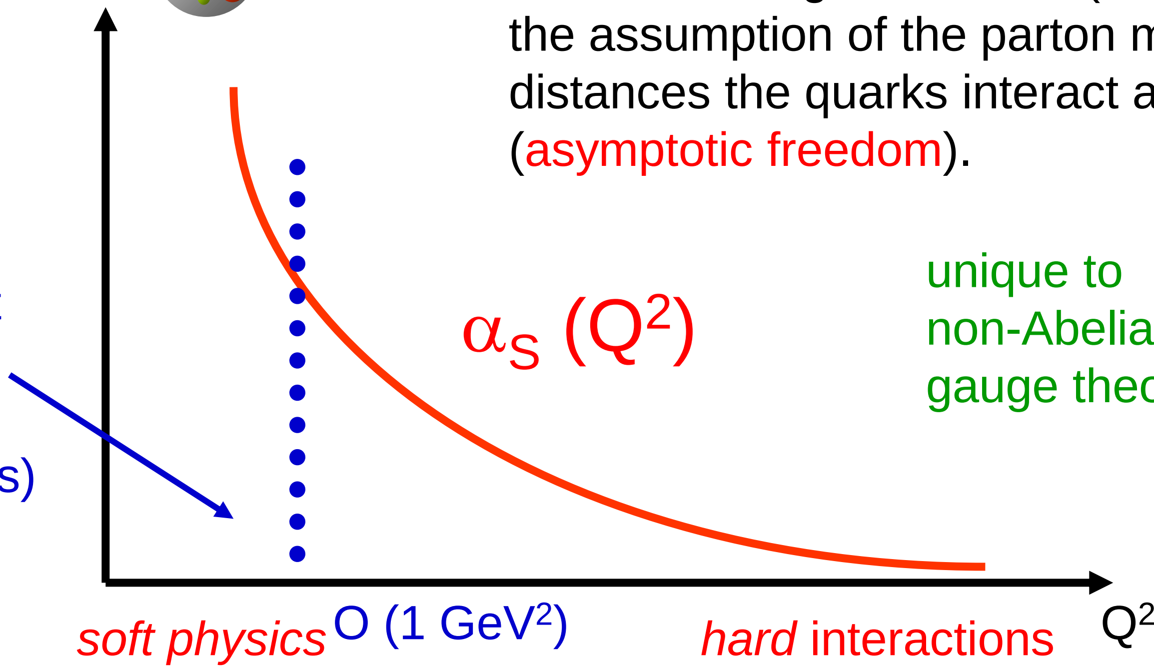
Let's write the asymptotic solution for $\alpha_s(Q^2)$ in terms of Λ_{QCD} . At leading order in QCD

$$\alpha\left(Q^2\right) = \frac{1}{b_0\log\left(Q^2/\Lambda^2_{QCD}\right)} = \frac{12\pi}{\left(33-2n_f\right)\log\left(Q^2/\Lambda^2_{QCD}\right)}$$

Running of $\alpha_S(Q^2)$

QCD reconciles quarks completely confined in hadrons at large distances (confinement) with the assumption of the parton model that at short distances the quarks interact almost freely (asymptotic freedom).





confinement of color (hadrons – color singlets)

unique to non-Abelian gauge theory

non-perturbative QCD
lattice QCD
chiral Lagrangian

perturbative QCD

However, most of the experimental support for QCD comes from comparisons with predictions which include higher-order QCD corrections: in the end α_S is not that small even at the highest energies achieved and higher-order correction are not negligible.

Asymptotic freedom

- coupling smaller at higher energies (smaller distances)
theory becomes effectively free

- a consequence of the sign of the beta function

- perturbation theory predicts asymptotic freedom

Confinement

- related to the fact that the coupling increases at small energies

- the behavior is still theoretically unknown because perturbation theory breaks down

- we do not have a rigorous explanation of confinement

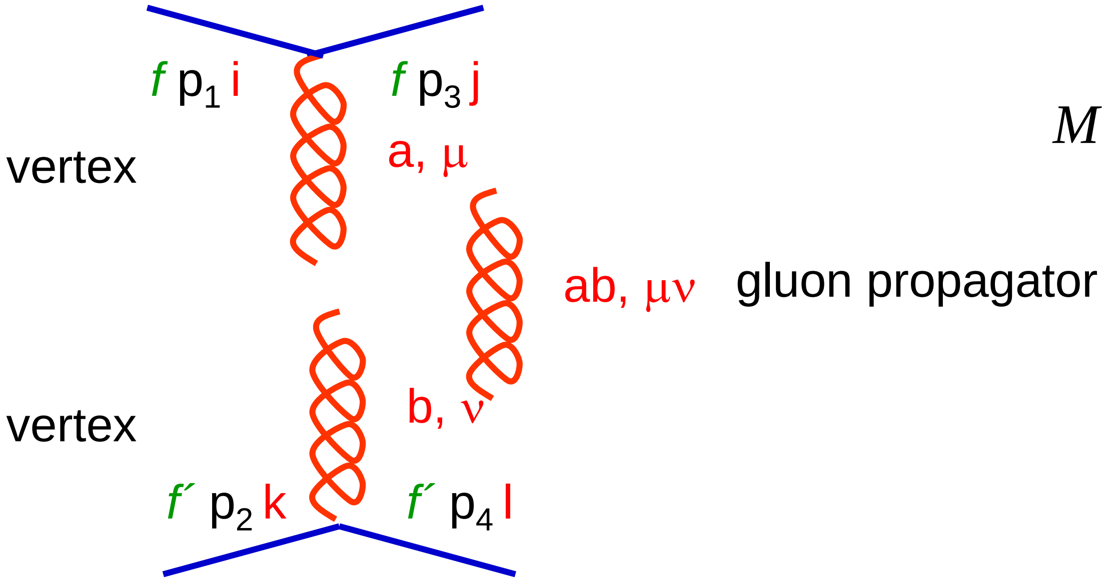
- we just observe that all partons are confined into color singlet hadrons:
if one tries to separate partons it becomes energetically favorable to extract from the QCD vacuum $q\bar{q}$ pairs and create new hadrons

- lattice QCD

- we assume that confinement always holds; proof worth an other Nobel Prize 20xx

Where calculations can be performed, QCD provides a very good and accurate description of relevant experimental data.

qq´ → qq´ Scattering



vertex factor

$$M_t = \left[\bar{u}_3^{\,j} \left(-i g_S \gamma^\mu \frac{\lambda_{ij}^a}{2} \right) u_1^i \right] \times$$

$$\frac{-i g_{\mu\nu} \delta_{ab}}{t} \times$$

$$\left[\bar{u}_4^{\,l} \left(-i g_S \gamma^\nu \frac{\lambda_{kl}^b}{2} \right) u_2^k \right]$$

the propagator imposes the same color δ_{ab} and same helicity $g_{\mu\nu}$ to the exchanged gluon at the interaction vertices

f, f´ quark flavors (i.e. ud → ud)
i, j, k, l quark colors
a, b, c gluon color combinations

M_t is the amplitude for a transition between an initial and final state with well defined spin and color.

$$M_t = i g_S^2 \frac{\lambda_{ij}^a}{2} \frac{\lambda_{kl}^a}{2} \left[\bar{u}_3^j \gamma^\mu u_1^i\right] \frac{1}{t} \left[\bar{u}_4^l \gamma_\mu u_2^k\right]$$

with u1, u2, u3, u4 the quarks spinors

the invariant amplitude is the same as in QED except for

the color factor $\frac{\lambda_{ij}^a}{2} \frac{\lambda_{kl}^a}{2}$

We proceed as in QED for eμ → eμ scattering:

if we do not observe the colors and do not measure the spins

1. average over initial colors, sum over final (always! the color is not observable)
2. average over initial spins, sum over final spins

$$\left\langle |M_t|^2 \right\rangle = \frac{1}{3\times 3} \sum_{colors} \frac{1}{2\times 2} \sum_{spins} |M_t|^2$$

The sum over spins gives

$$\sum_{spins} |M_t|^2 = g_S^4 \sum_{s_1,s_2,s_3,s_4} \left| \bar{u}_3^j \gamma^\mu u_1^i \ \frac{1}{t} \ \bar{u}_4^l \gamma_\mu u_2^k \right|^2 =$$

$$32 \frac{g_S^4}{t^2} \left[(p_3 \cdot p_4)\cdot(p_1 \cdot p_2) + (p_3 \cdot p_2)\cdot(p_4 \cdot p_1) \right]$$

$$\sum_{spins} |M_t|^2 = 8 g_S^4 \frac{\left(s^2 + u^2\right)}{t^2}$$

and the sum over colors gives

$$\sum_{colors}\cdots=\sum_{a,b,i,j,k,l}\left(\frac{\lambda^a_{ij}}{2}\frac{\lambda^a_{kl}}{2}\right)^+\left(\frac{\lambda^b_{ij}}{2}\frac{\lambda^b_{kl}}{2}\right)=\sum_{a,b}\sum_{i,j}\left(\frac{\lambda^a_{ij}}{2}\frac{\lambda^b_{ji}}{2}\right)\cdot\sum_{k,l}\left(\frac{\lambda^a_{kl}}{2}\frac{\lambda^b_{lk}}{2}\right)=$$
$$=\sum_{a,b}Tr\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right)Tr\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right)=2$$

Averaging over initial colors (3 × 3) and spins (2 × 2) we finally obtain

$$\left\langle |M_t|^2\right\rangle = g_S^4\,\frac{2}{9}\,\frac{8}{4}\,\frac{\left(s^2+u^2\right)}{t^2} \quad\Rightarrow\quad \frac{d\sigma}{dt}=\frac{1}{16\pi s^2}\,16\pi^2\alpha_S^2\,\frac{2}{9}\,2\,\frac{\left(s^2+u^2\right)}{t^2}$$

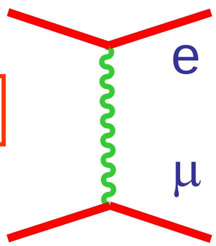
color factor (1/3×1/3×2)

spin factor (1/2×1/2×8)

flux and phase space

$$\alpha_S=\frac{g_S^2}{4\pi}$$

QED



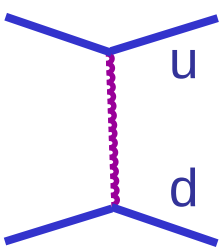
$$\sum_{\text{spins}}|\mathcal{M}|^2 = 8\,e^4\,\frac{s^2+u^2}{t^2}$$

spin and color averaging:

QED: 1/2 × 1/2 = 1/4

QCD: 1/2 × 1/2 × 1/3 × 1/3 = × 1/36

QCD



$$\sum_{\text{spins, colours}}|\mathcal{M}|^2 = 8\,g_s^4\,\frac{s^2+u^2}{t^2}\times \text{CF}$$

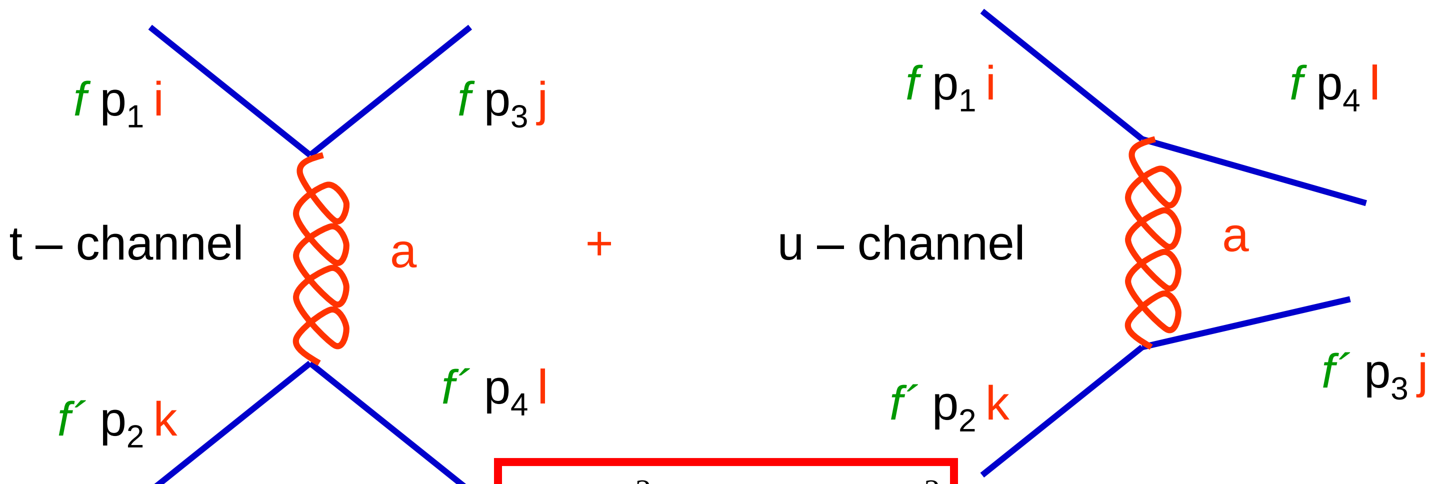
In summary, the invariant matrix element for $q_\alpha q_\beta \to q_\alpha q_\beta$, with $\alpha \neq \beta$ is given by

$$\langle |M_t|^2 \rangle = g_S^4 \frac{2}{9} 2 \frac{(s^2+u^2)}{t^2}$$

Since the gluon is a singlet with respect to flavor (i.e. cannot change the flavor of interacting quarks) and all quantum numbers at the vertex are conserved, the u – channel cannot be present for the interactions of two quarks of different flavor.

For $q_\alpha \bar{q}_\beta \to q_\alpha \bar{q}_\beta$, with $\alpha \neq \beta$ one obtains the same matrix element.

For $q_\alpha q_\beta \to q_\alpha q_\beta$ with $\alpha = \beta$ (identical quarks, e.g. ss $\to$ ss) one has to consider also the u – channel (crossing 3 $\leftrightarrow$ 4)



$$|M_{t+u}|^2 = |M_t + M_u|^2$$

To calculate the invariant amplitude M_u for the u – channel
we proceed in the same way as for M_t with the exchange 3 $\leftrightarrow$ 4 (crossing)

$$M_u = i g_S^2 \frac{\lambda_{il}^b}{2} \frac{\lambda_{kj}^b}{2} \left[\bar{u}_3^j \gamma^\mu u_2^k\right] \frac{1}{u} \left[\bar{u}_4^l \gamma_\mu u_1^i\right]$$

Before averaging over initial spin and color states and summing over final spin and color states, however, we have to add the two amplitudes …

$$M_t + M_u$$

The invariant matrix element for $q_\alpha q_\beta \rightarrow q_\alpha q_\beta$, with $\alpha = \beta$ is given by

$$\left\langle \left|M_{t+u}\right|^2 \right\rangle = g_S^4 \frac{2}{9}\left[2\frac{\left(s^2+u^2\right)}{t^2} + 2\frac{\left(t^2+s^2\right)}{u^2} - \frac{1}{3}\frac{4s^2}{tu}\right]$$

The last term comes from the interference of the t – and the u – channels.

For $q_\alpha \bar{q}_\beta \rightarrow q_\alpha \bar{q}_\beta$, with $\alpha = \beta$, we have the t – and s – channels: $M_t + M_s$

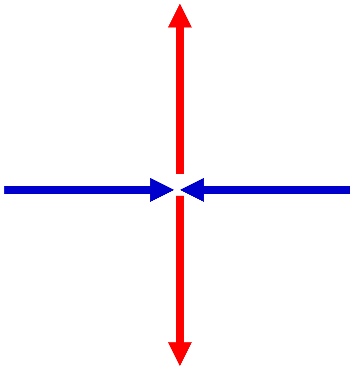
$$\left\langle \left|M_{t+s}\right|^2 \right\rangle = g_S^4 \frac{2}{9}\left[2\frac{\left(s^2+u^2\right)}{t^2} + 2\frac{\left(t^2+u^2\right)}{s^2} - \frac{1}{3}\frac{4u^2}{st}\right]$$

t – and s – channel interference term

QCD 1 + 2 → 3 + 4 Process

$$\frac{d\hat{\sigma}}{d\hat{t}}(1+2\to 3+4)=\frac{1}{16\pi\ \hat{s}^2}\left\langle\left|M(1+2\to 3+4)\right|^2\right\rangle=\frac{\pi\alpha_S^2}{\hat{s}^2}|A|^2$$

Process	$\lvert A\rvert^2$	*strength* at 90^0 in c.o.m.
$q_1q_2\to q_1q_2,\ q_1\bar{q}_2\to q_1\bar{q}_2$	$\frac{4}{9}\frac{s^2+u^2}{t^2}$	2.22
$q_1q_1\to q_1q_1$	$\frac{4}{9}\left(\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}\right)-\frac{8}{27}\frac{s^2}{ut}$	3.26
$q_1\bar{q}_1\to q_2\bar{q}_2$	$\frac{4}{9}\frac{t^2+u^2}{s^2}$	0.22
$q_1\bar{q}_1\to q_1\bar{q}_1$	$\frac{4}{9}\left(\frac{s^2+u^2}{t^2}+\frac{t^2+u^2}{s^2}\right)-\frac{8}{27}\frac{u^2}{st}$	2.59
$q\bar{q}\to gg$	$\frac{32}{27}\frac{u^2+t^2}{ut}-\frac{8}{3}\frac{u^2+t^2}{s^2}$	1.04
$gg\to q\bar{q}$	$\frac{1}{6}\frac{u^2+t^2}{ut}-\frac{3}{8}\frac{u^2+t^2}{s^2}$	0.15
$qg\to qg$	$-\frac{4}{9}\frac{u^2+s^2}{us}+\frac{u^2+s^2}{t^2}$	6.11
$gg\to gg$	$\frac{9}{2}\left(3-\frac{ut}{s^2}-\frac{us}{t^2}-\frac{st}{u^2}\right)$	30.4



The coefficient 1 / (16 π s^2) is the phase space and flux factor.

For Next Week

Study the material and prepare / ask questions
Study ch. 14 (sec. 3, 4) and ch. 2 (sec.15) in Halzen & Martin
and / or ch. 10 in Thomson

Do the homeworks

Next week we will study the QCD parton model
refresh the parton model, ch. 8 and 9 in Halzen and Martin
have a first look at the lecture notes, you can already have questions
read ch. 10 (sec. 1 to 8) in Halzen & Martin and / or ch. 10 (sec. 6) in Thomson