Advanced Particle Physics 2

Strong Interactions

and Weak Interactions

L5 – $e^+e^- \to q\bar{q}$ Annihilation

(http://dpnc.unige.ch/~bravar/PPA2/L5)

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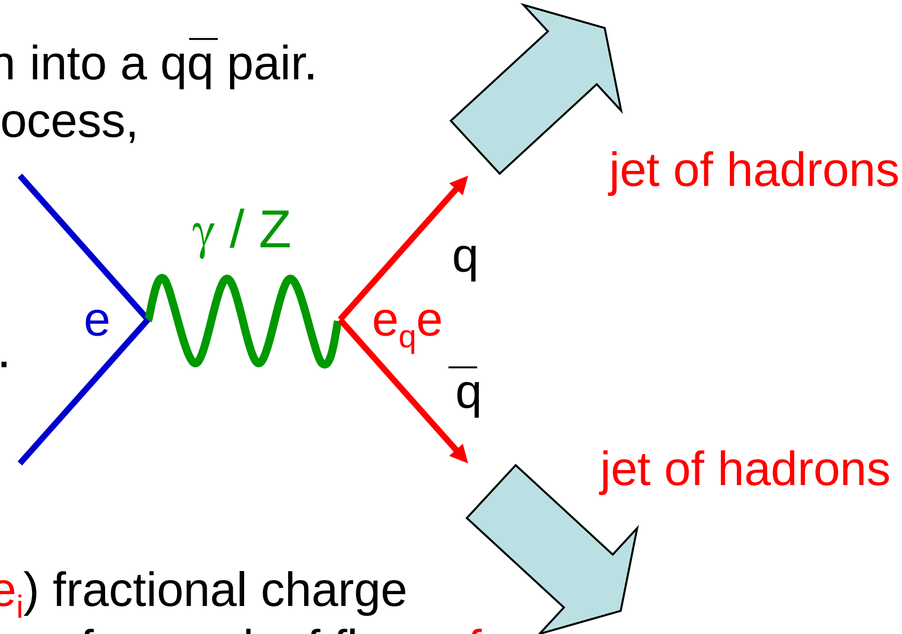
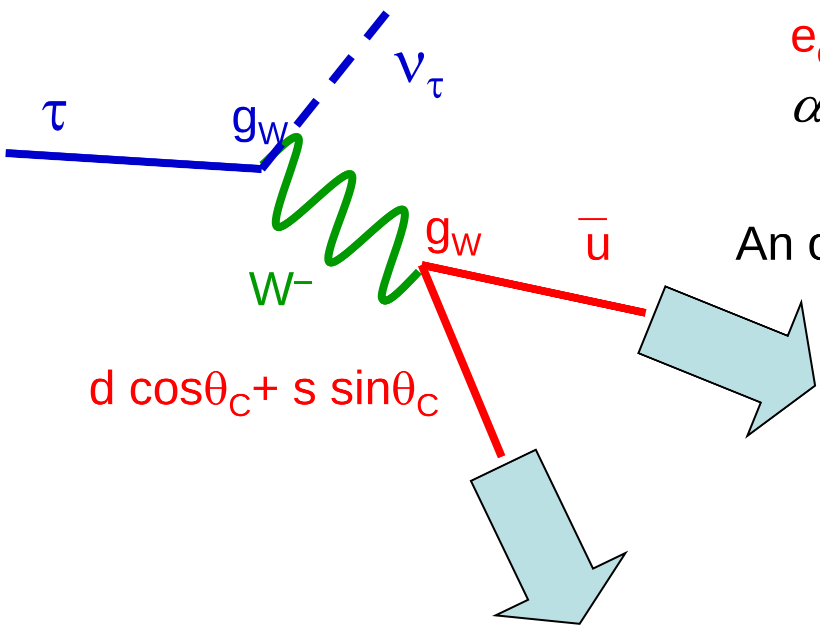
QCD Is Not Only Proton Structure!

Consider, for example, the e^-e^+ annihilation into a $q\bar{q}$ pair.
While the annihilation is an electroweak process, the subsequent evolution of quarks and their "transition" into observable hadrons – fragmentation – is controlled by the strong interaction – QCD.
By studying this, and similar processes we can learn a big deal about QCD.



note
e_q (e_f, e_i) fractional charge of a quark of flavor *f*
$e_q e$ charge of a quark

$$\alpha_{EM} \rightarrow e_q^2 \alpha_{EM}$$



An other example is the τ lepton decay into hadrons.

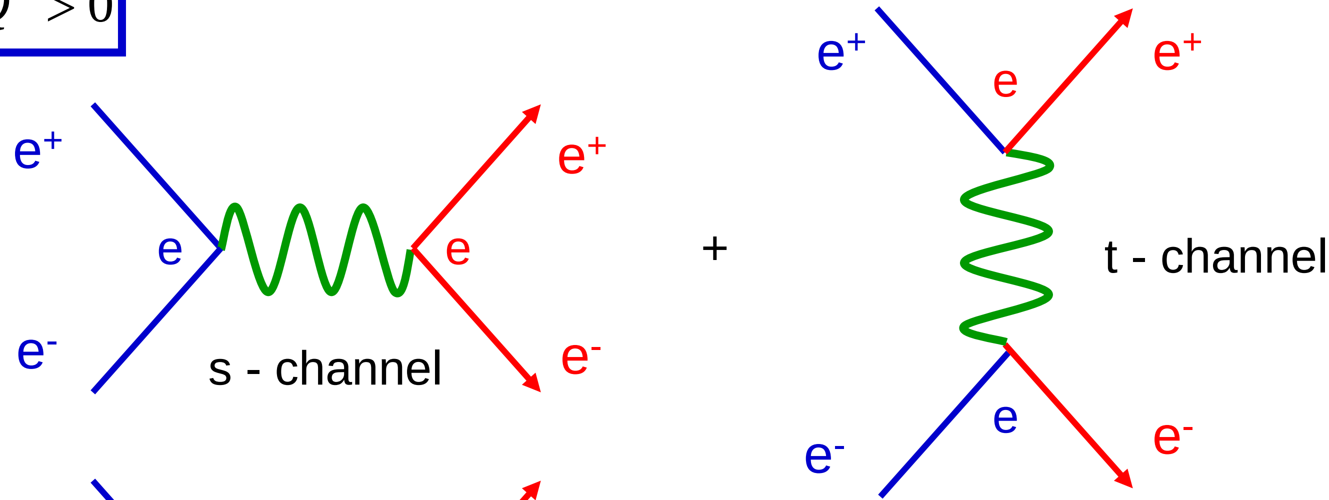
In general, when the final state involves hadrons, strong interactions are at play and can mask the underlying weak or electromagnetic sub-process.

$e^+ e^- \rightarrow q\,\bar{q}$ Annihilation

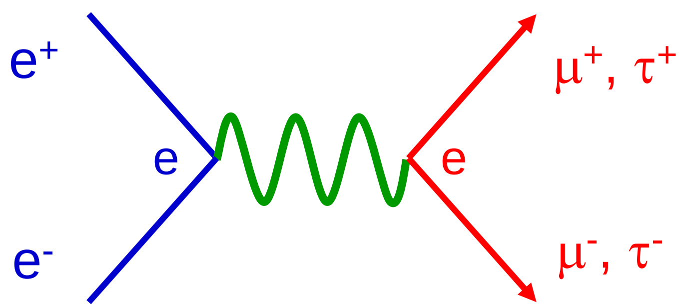
Possible processes in the $e^- e^+$ interaction:

$$s = \left(E_{e-} + E_{e+}\right)^2 = Q^2 > 0$$

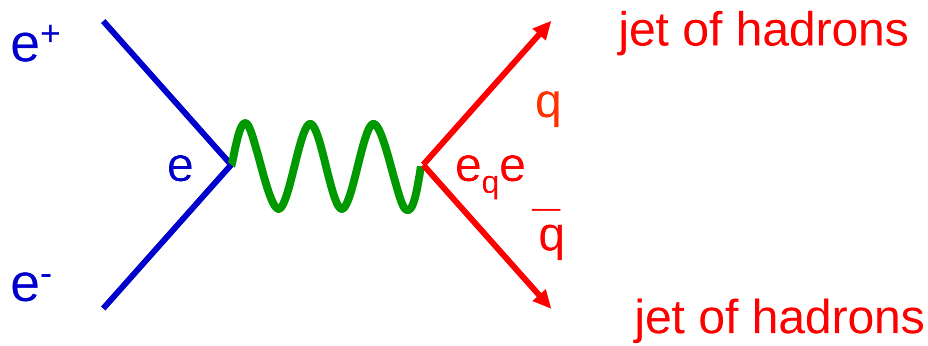
$e^+ e^- \rightarrow e^+ e^-$



$e^+ e^- \rightarrow \mu^+ \mu^-$



$e^+ e^- \rightarrow q\,\bar{q}$



The Process $e^+ e^- \to q\,\bar{q}$

High resolution photons can be prepared by colliding high energy e^- and e^+ beams (note that $m_{\gamma^*} > 0$ – timelike photon to be compared to DIS where $m_{\gamma^*} < 0$ – spacelike photon). $e^- e^+$ colliders can be used to study QED, (electro)weak interactions, quarks, gluons and QCD, or search for heavy quarks and leptons, or new particles, … however the quantum numbers of the final state must be that of the γ^*: $J^{PC} = 1^{--}$.

The $e^- e^+ \to q\,\bar{q}$ process is described by the same Feynman diagram as the $e^- e^+ \to \mu^+ \mu^-$ (s – channel only), where one replaces the $\mu^+ \mu^-$ pair by a $q\bar{q}$ pair assuming that the quarks are Dirac particles (i.e. that they have spin ½)

In addition, we have to take into account the masses and electric charges of quarks, which are different and the color charge that only quarks carry.

The calculation of the Feynman diagrams (neglecting masses) for $e^+ e^- \to \mu^+ \mu^-$ gives

$$\left.\frac{d\sigma^{e+e-\to\mu+\mu-}}{d(\cos\theta)}\right|_{cm} = \frac{\alpha^2}{2s}\left[\frac{u^2+t^2}{s^2}\right] = \frac{2\pi\alpha^2}{4s}\left(1+\cos^2\theta\right) \quad \to \quad \sigma_{tot}^{e+e-\to\mu+\mu-} = \frac{4\pi}{3s}\alpha^2$$

and for $q\bar{q}$ production becomes

$$\left.\frac{d\sigma^{e+e-\to q\bar{q}}}{d(\cos\theta)}\right|_{cm} = C_F e_q^2 \frac{\alpha^2}{2s}\left[\frac{u^2+t^2}{s^2}\right] = C_F e_q^2 \frac{2\pi\alpha^2}{4s}\left(1+\cos^2\theta\right) \to \sigma_{tot}^{e+e-\to q\bar{q}} = C_F e_q^2 \frac{4\pi}{3s}\alpha^2$$

with $e_q\, e$ the electric charge of quarks and $C_F = 3$ the color factor.

In deriving the $q\bar{q}$ cross section we have to pay attention to the fact that the $q\bar{q}$ pair is produced in a color singlet state!
[note that a "colorless" state is not necessarily invariant under SU(3) transformations]

If this were the case (color eigen-state, i.e. $R\bar{R}$)

$$\sigma_{tot}^{e+e-\to q\bar{q}} \sim \sum_{color\ i}^{3} \left| \left\langle q_i \bar{q}_i \middle| e^+ e^- \right\rangle \right|^2 = 3e_q^2 \left| \left\langle \mu^+ \mu^- \middle| e^+ e^- \right\rangle \right|^2 \approx 3e_q^2 \sigma_{tot}^{e+e-\to\mu+\mu-}$$

Although the result is the expected one, the reasoning behind it is not correct!

According to QCD, the $q\bar{q}$ pair is created in a color singlet state:

$$\left|(q\bar{q})_S\right\rangle = \frac{1}{\sqrt{3}}\left(\left|q_R\bar{q}_R\right\rangle + \left|q_G\bar{q}_G\right\rangle + \left|q_B\bar{q}_B\right\rangle\right)$$

and

$$\sigma_{tot}^{e+e-\to q\bar{q}} \sim \left|\left\langle (q\bar{q})_S \middle| e^+e^-\right\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{3}}\left\langle q_R\bar{q}_R \middle| e^+e^-\right\rangle + \frac{1}{\sqrt{3}}\left\langle q_B\bar{q}_B \middle| e^+e^-\right\rangle + \frac{1}{\sqrt{3}}\left\langle q_G\bar{q}_G \middle| e^+e^-\right\rangle\right|^2 = \frac{1}{3}e_q^2\left|3\left\langle \mu^+\mu^- \middle| e^+e^-\right\rangle\right|^2$$

$$= 3e_q^2\sigma_{tot}^{e+e-\to\mu+\mu-}$$

Observation of Quark Jets

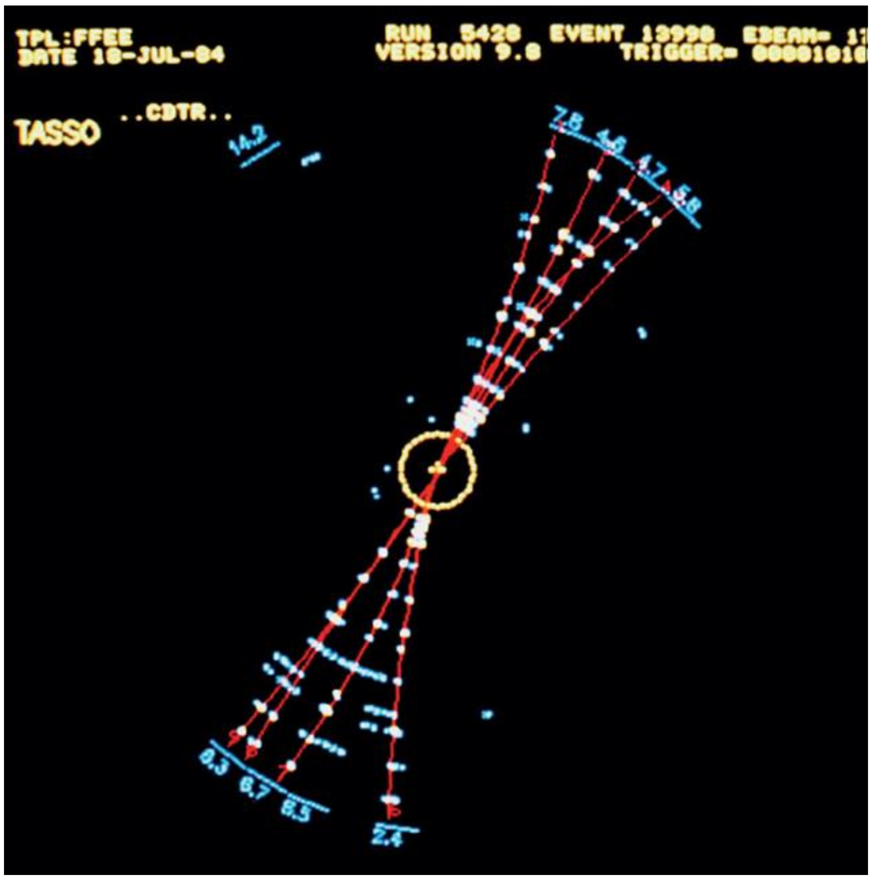
Jet = collimated spray of hadrons from quark or gluon fragmentation / hadronization

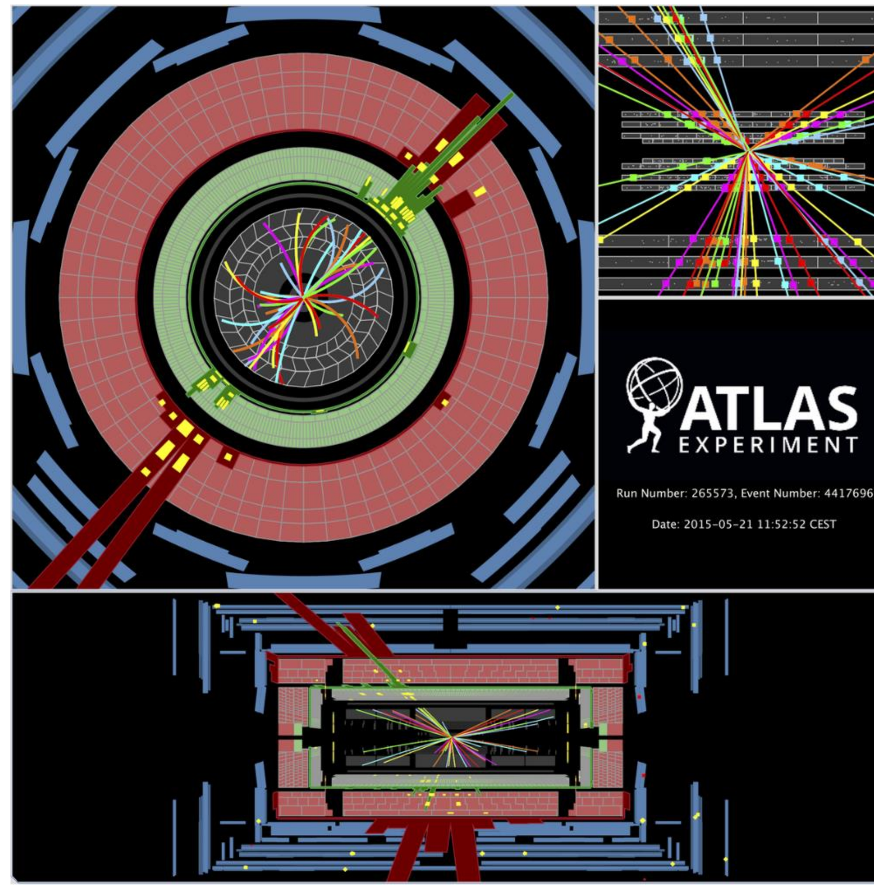
$$e^+ \; e^- \rightarrow \text{jet}_1 + \text{jet}_2$$

$$\left(e^+ \; e^- \rightarrow q \; \bar{q}\right)$$

$$p \; p \rightarrow \text{jet}_1 + \text{jet}_2 + X$$

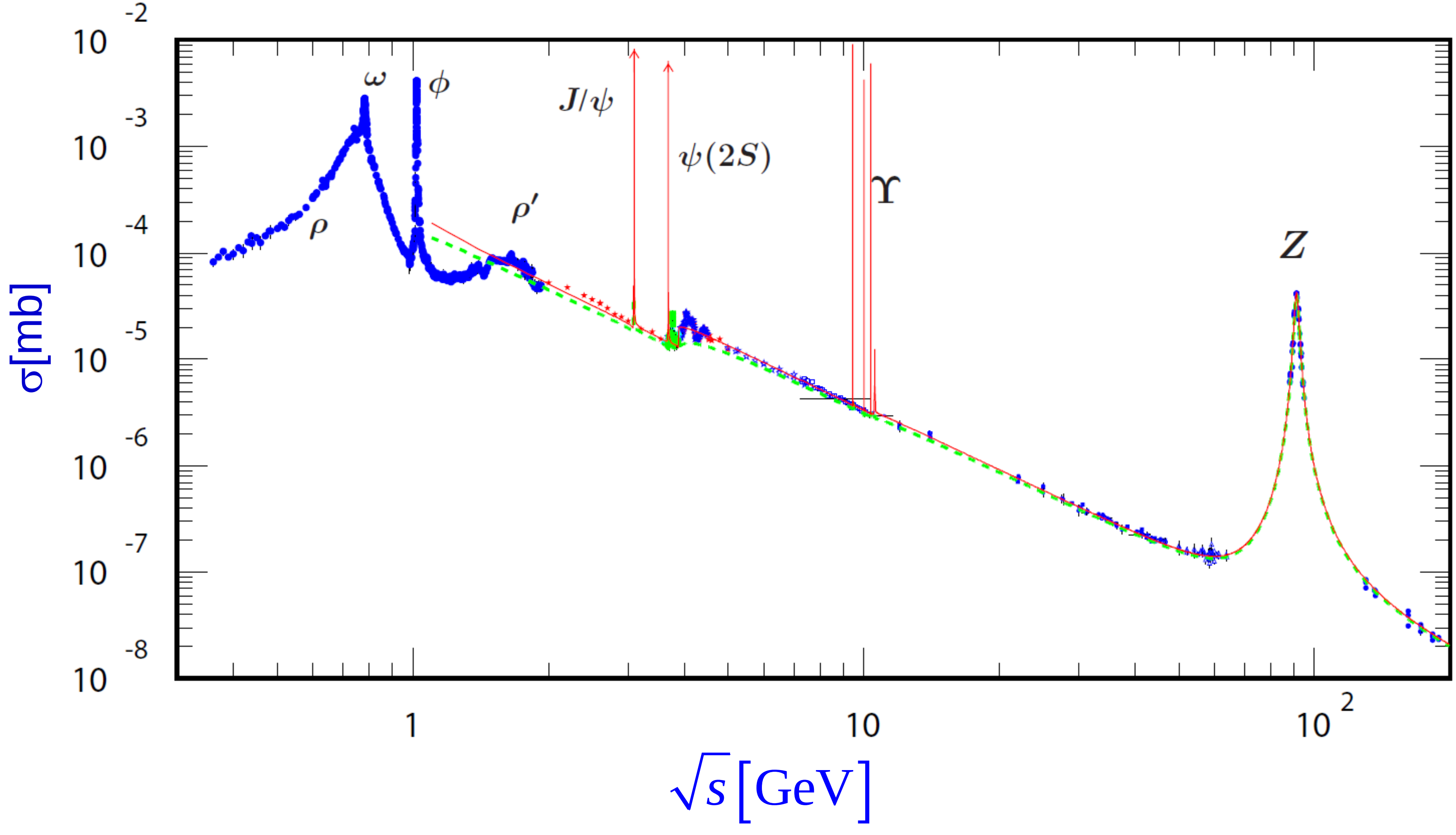
$$\left(q \; \bar{q} \rightarrow q \; \bar{q}\right)$$





To “see” jets, need quarks with sufficient energy.

$e^+ e^- \to$ hadrons



In the total $e^+ e^-$ annihilation cross section → hadrons, σ^{tot}, all quarks with $m_q < \sqrt{s}/2$ enter. It is given by the sum of σ for different quark flavor accessible at a given c.o.m. energy s:

flavor	d	u	s	c	b	t
el. charge [e]	-1/3	2/3	-1/3	2/3	-1/3	2/3
mass [GeV]	~0.005	~0.003	0.15	1.35	4.5	174

$$\sigma\left(e^+e^- \to hadrons\right) = \sum_f \sigma\left(e^+e^- \to q\bar{q}\right) = 3\sum_f \sigma\left(e^+e^- \to \mu\bar{\mu}\right)$$

This expression describes well the behavior of σ^{tot} outside of the resonance regions. This simple calculation leads to the prediction

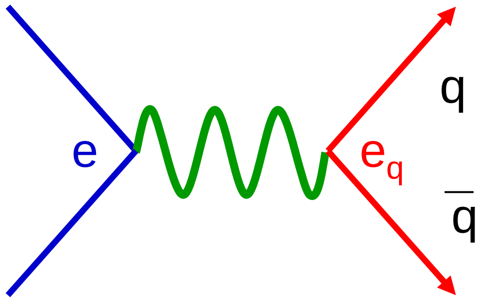
$$R = \frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\sum_f e_f^2$$

(the number of terms in the sum – the number of active flavors *f* – depends on s)
The total $e^+ e^-$ annihilation cross section into hadrons therefore directly counts the number of quarks, their flavors, and their colors.

	no color	3 colors
for u, d, s active flavors	R = 2 /3	R = 2
for u, d, s, c	R = 10 / 9	R = 10 / 3
for u, d, s, c, b	R = 11 / 9	R = 11 / 3

The result will be modified when interpreted in the context of QCD (gluon radiation). Note that data support the idea that there are only 3 colors!

The Ratio *R*

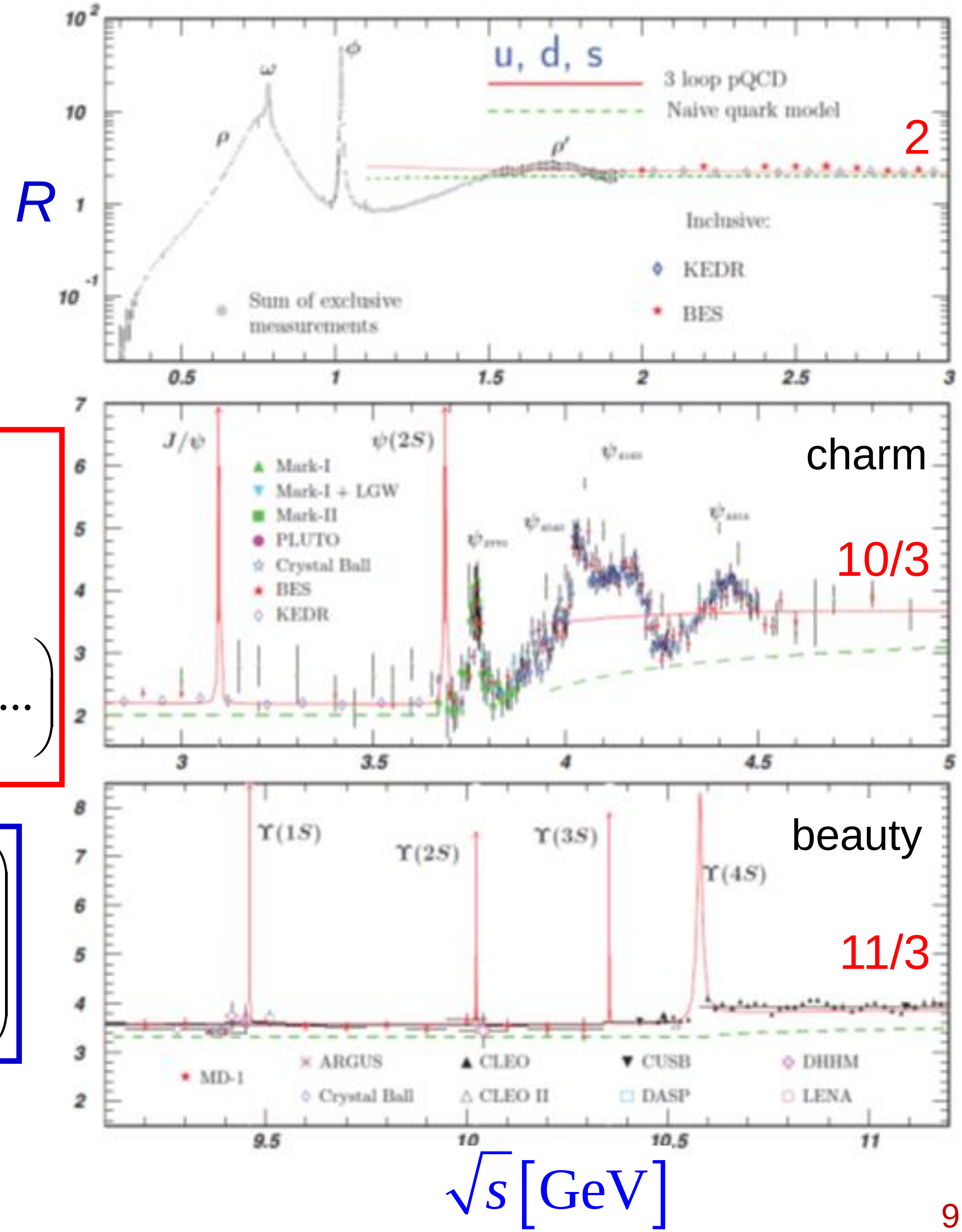


$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_C \frac{\sum_f e_f^2 \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} \left(1 + \frac{\alpha_S}{\pi} + \ldots\right)$$

$$R \approx N_C \times \begin{pmatrix} 2/3 & (uds) \\ 10/9 & (udsc) \\ 11/9 & (udscb) \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 10/3 \\ 11/3 \end{pmatrix}$$

→ fractional charges for quarks

→ 3 colors

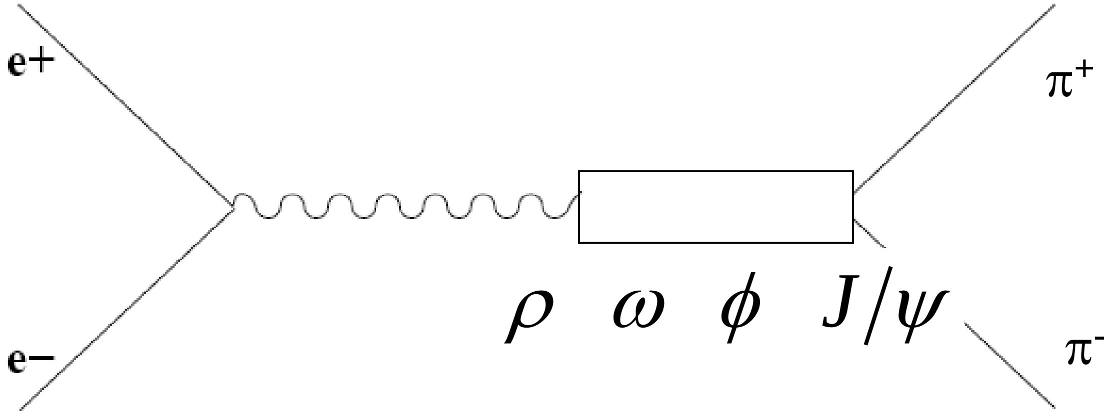


The Resonances

According to the Vector Meson Dominance (VDM) model of Sakurai
a photon ($J^{PC} = 1^{--}$) can fluctuate to a vector meson, like ρ^0, ω, ϕ, or J/Ψ ($J^{PC} = 1^{--}$).
If the c.o.m. energy is that of the resonance, instead of fluctuating back to a photon, the vector meson can materialize and "decays".

Moreover, in DIS the vector meson can interact with a proton target as a hadron.
This explain roughly why $\sigma_{\gamma p} \sim \alpha\, \sigma_{hadronic}$ (i.e. 100 $\times$ smaller).

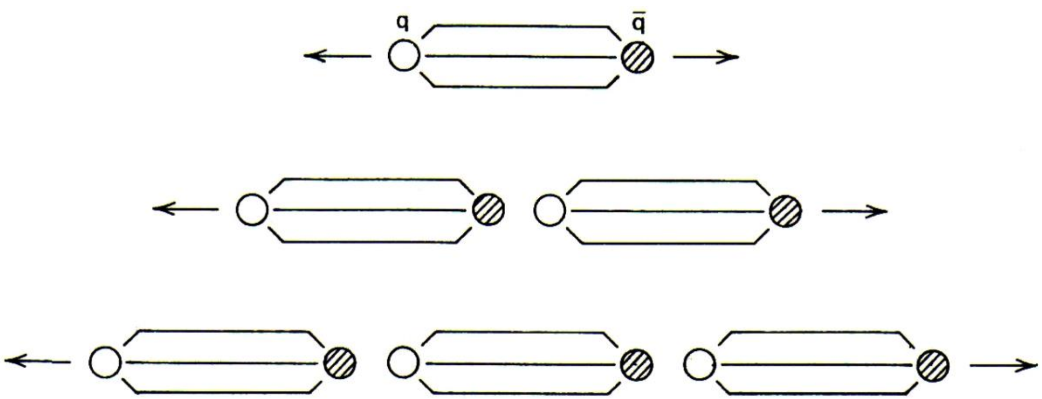
$$e^+e^- \to \gamma^* \to \text{resonance } (J^{PC} = 1^{--}) \to \begin{cases} l\bar{l}\,(+\gamma) \\ \text{hadrons}(+\gamma) \end{cases}$$



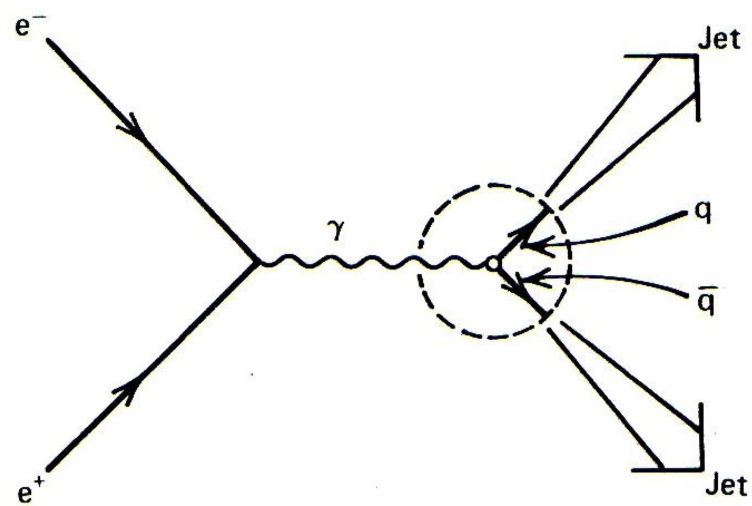
Hadronization

Because of confinement, we cannot observe directly the quarks (so far, particles with fractional charges have never been observed). In a simplified picture, once created, the $q\bar{q}$ pair separates with equal and opposite momenta and materializes (hadronizes) into two back-to-back jets of colorless hadrons, which conserve the essential kinematical properties of initial quarks. This process is called hadronization.

The hadronization mechanism is not yet fully understood.
Note that quarks must fragment into hadrons with unit probability.
We can visualize it as a cascade of $q\bar{q}$ pairs created as the initial quarks separate.
Hadrons in a jet are collimated around the initial quark direction with $\langle p_T \rangle \sim 300$ MeV.
The fragmentation cone narrows very slowly as $\langle\theta\rangle \sim 1 / \log Q^2$.

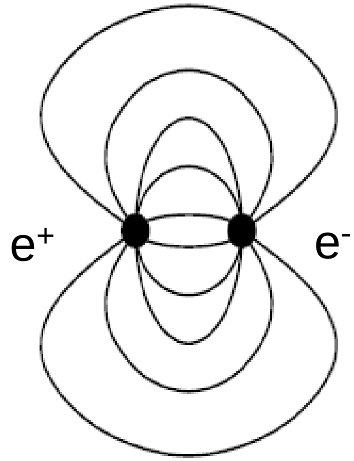


at 1 fm separation $E_{string} \sim 1$ GeV

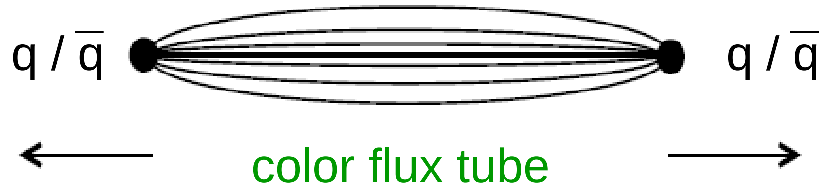


A similar picture can be applied also to jet production in deep inelastic scattering.

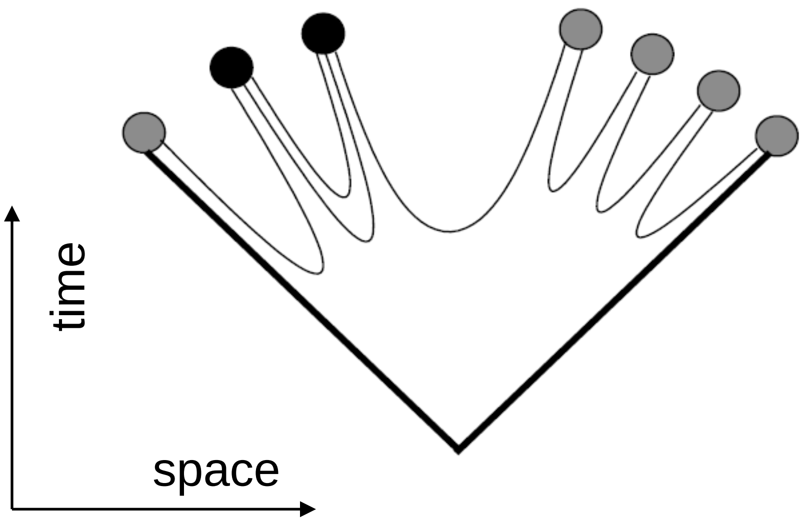
strong force at very short distance r << 1 fm, V ~ 1/r



strong force at long distances r > 1 fm, V ~ kr



time evolution



hadronization

charge, baryon number, strangeness, etc. must be conserved !

the distribution of hadrons around the initial quark direction (jet) is approximately gaussian with

$$\frac{d\sigma}{dp_T^2} \sim \exp\left(-\sigma_t p_T^2\right)$$

$\sigma_t \sim 1.5\ \text{GeV}^{-2} \rightarrow \langle p_T \rangle \sim 300$ MeV
(parameter determined empirically)

Hadronization and Jets

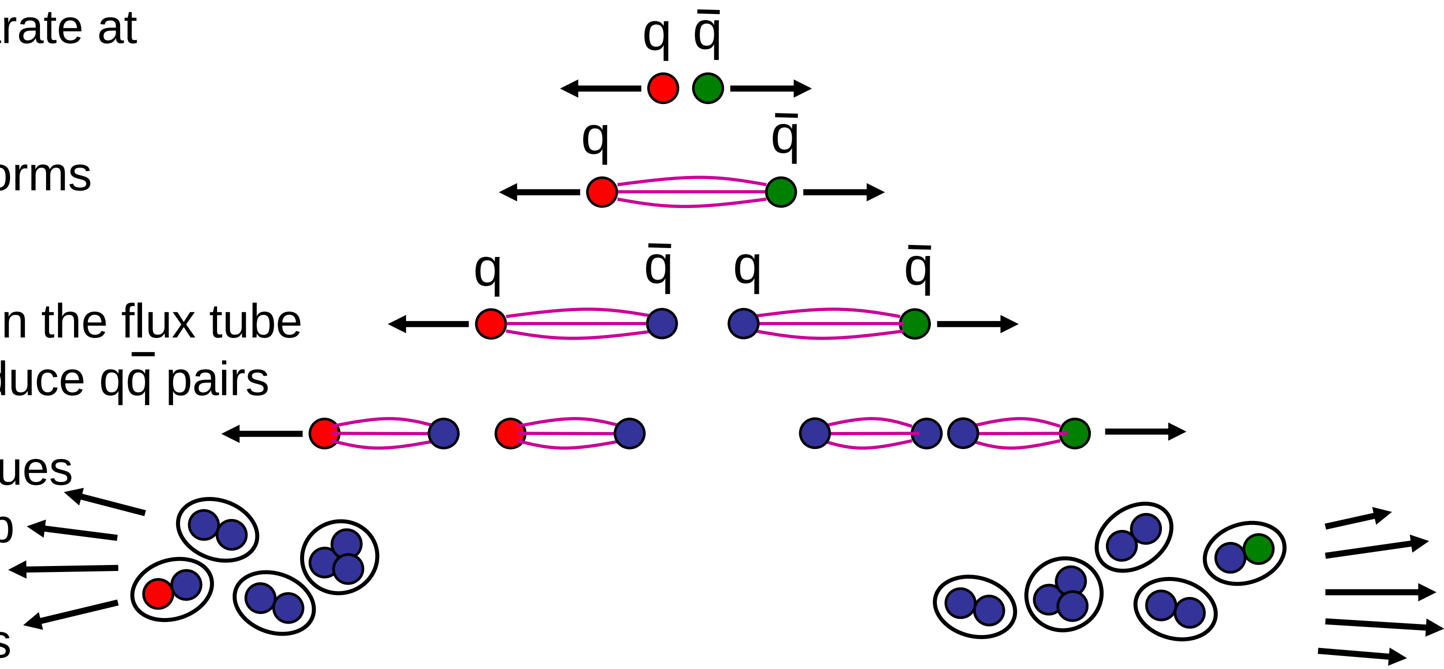
Consider a quark and anti-quark pair produced in electron positron annihilation:

i) initially quarks separate at high velocity

ii) a colour flux tube forms between quarks

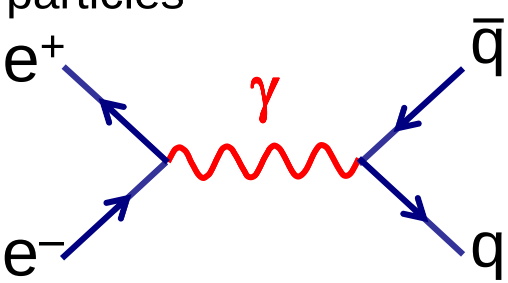
iii) the energy stored in the flux tube is sufficient to produce $q\bar{q}$ pairs

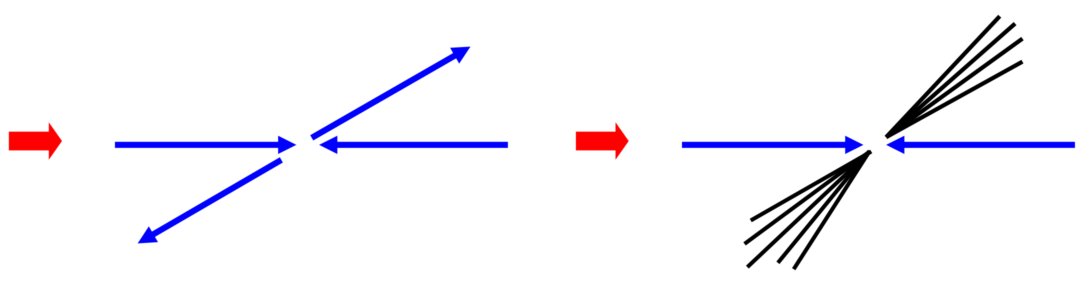
iv) the process continues until quarks pair up into jets of colourless hadrons



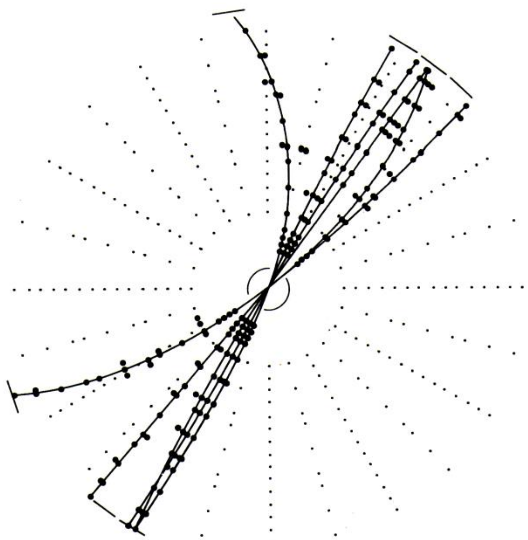
This process is called hadronization. It is not (yet) calculable.

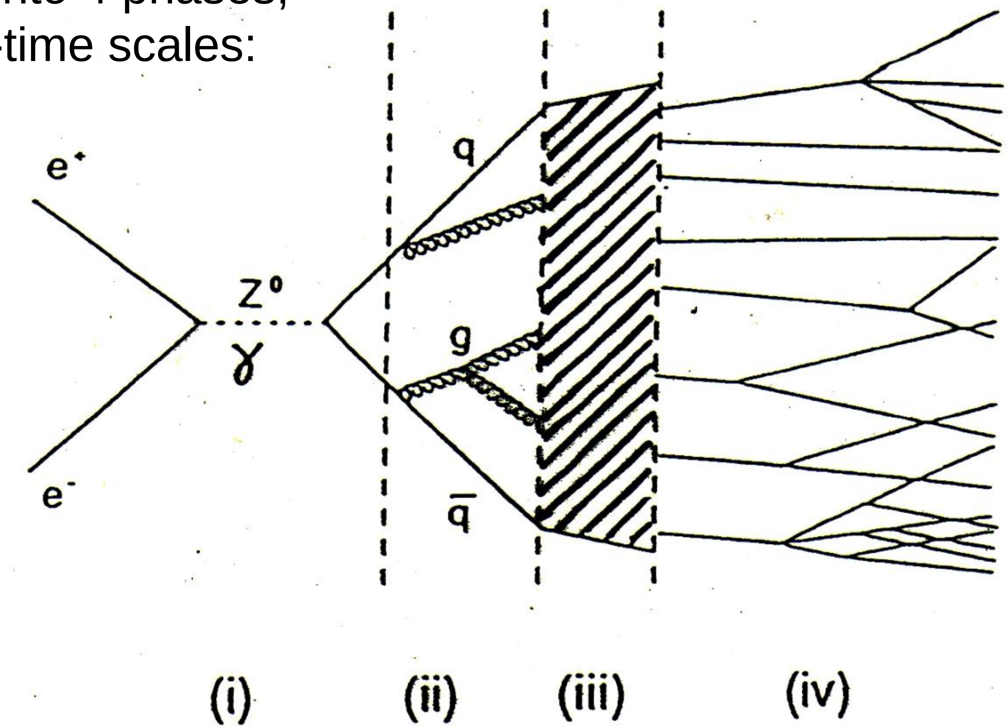
The main consequence is that at collider experiments quarks and gluons are observed as jets of particles





In the annihilation process $e^+ e^- \rightarrow$ hadrons,
we can try to separate the interaction into 4 phases,
which correspond to 4 different space-time scales:





(i) 10^{-17} cm – $q\bar{q}$ production (electroweak)
(ii) 10^{-15} cm – gluon radiation (perturbative QCD)
(iii) 10^{-13} cm – hadronization: fragmentation of quarks and gluons into hadrons (non-perturbative QCD)
(iv) $>10^{-13}$ cm – decays of unstable particles (electroweak)

To reproduce the characteristics of the events we use sophisticated Monte Carlo event generators which simulate the hadronization as a stochastic process.
Typically, there are ~100 parameters describing the hadronization process.

Jets

To identify a jet, we have to define a jet axis and group particles around this axis with an algorithm that selects only those particles which belong to this particular jet. Let's examine two such algorithms / variables.

The thrust T defines an axis which maximizes the longitudinal momentum of hadrons in the jet w.r.t this axis (it corresponds to the initial quark direction).

$$T = \max\left[\frac{\sum_i p_L^i}{\sum_i \left|p^i\right|}\right]$$

T $\rightarrow$ 1 for a di-jet event
T $\rightarrow$ 0.5 for an istotropic event

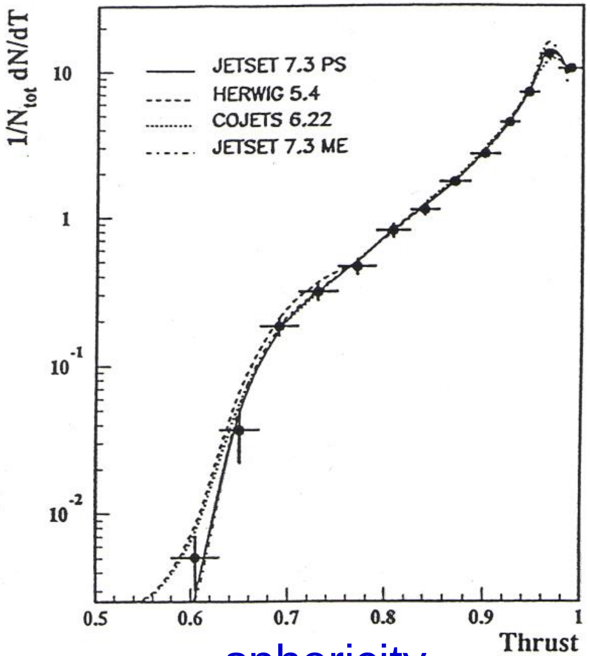
The sphericity S defines an axis that minimizes the transverse momentum of hadrons

$$S = \frac{3}{2}\min\left[\frac{\sum_i \left(p_T^i\right)^2}{\sum_i \left(p^i\right)^2}\right]$$

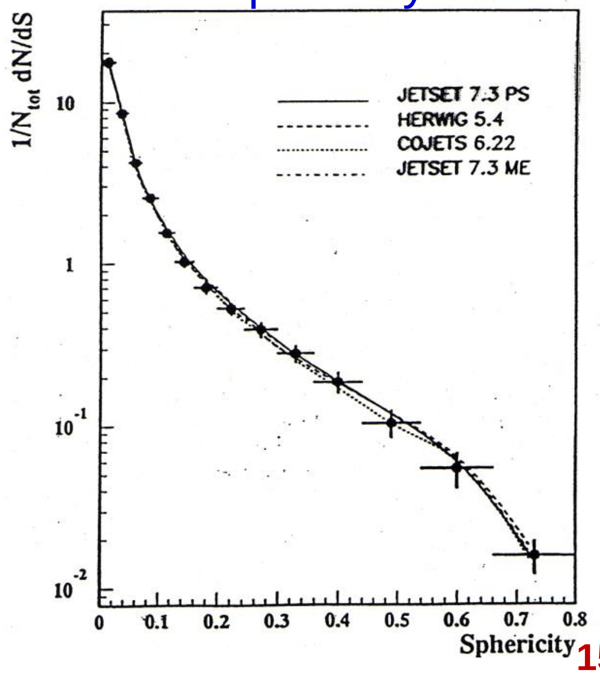
S $\rightarrow$ 1 for an isotropic event
S $\rightarrow$ 0 for a di-jet event

In principle, the two axes should coincide.

thrust

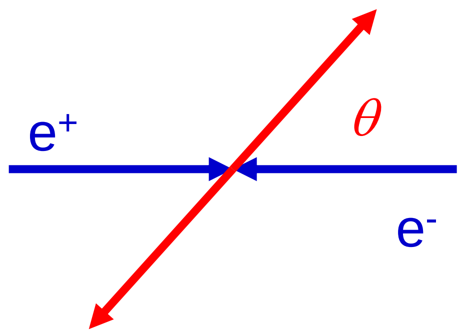


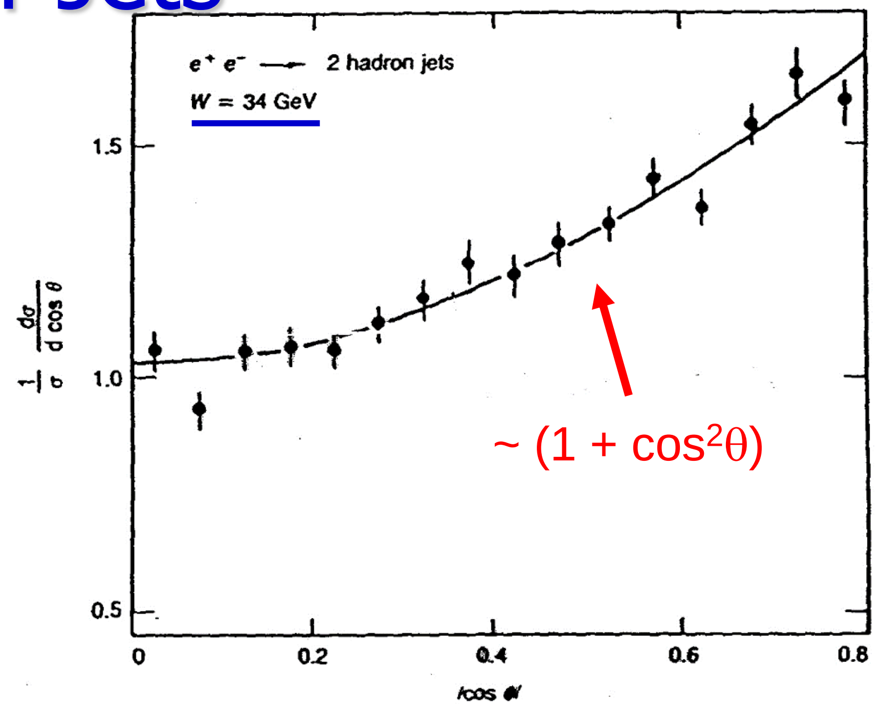
sphericity



Angular Distribution of Jets

Now that we have defined the jet axis (thrust) we can measure the angular distribution of the jet axis w.r.t. the $e^+ e^-$ collision axis.

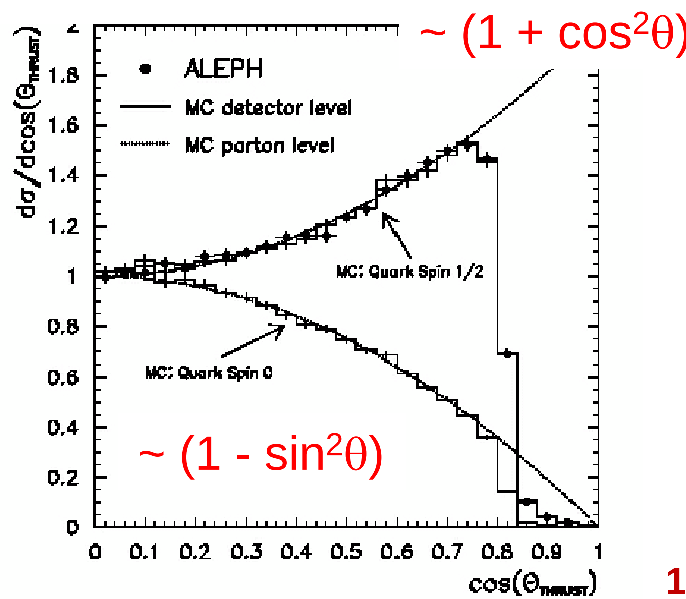




$$\frac{d\sigma}{d(\cos\theta)} = 3\sum_f e_f^2 \frac{2\pi\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

extra factors: 3 for colour

e_f fractional charge of quarks



The jet axis follows the same distribution $\propto (1 + \cos^2\theta)$ as the muons in the $e^+ e^- \to \mu^+ \mu^-$ process

This distribution is sensitive to quark's spin:

quarks have spin 1/2.

If the quark had spin 0, the distribution would be $\propto (1 - \sin^2\theta)$,

like in the non-resonant $e^+ e^- \to \pi^+ \pi^-$ process.

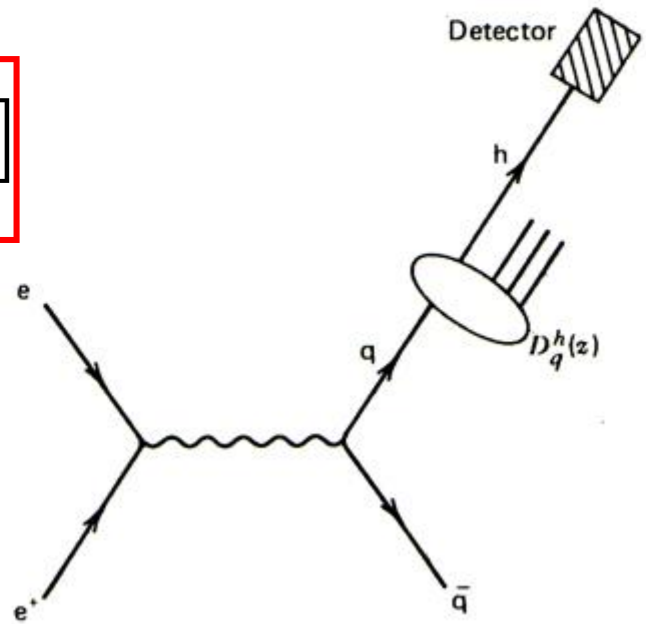
The Fragmentation Process

Even though we cannot calculate the fragmentation of quarks into hadrons from first principles, we can try to describe it, as we did for quarks inside hadrons.
We introduce the fragmentation functions $D_q^h(z)$, which represent the probability that a quark q will fragment to a hadron h with fractional momentum $z = E_h / E_q$.

The differential cross section can be factorized as

$$\frac{d\sigma\left(e^+e^- \to hX\right)}{dz} = \sum_q \sigma\left(e^+e^- \to q\bar{q}\right)\left[D_q^h(z) + D_{\bar{q}}^h(z)\right]$$

where $z \equiv \frac{E_h}{E_q} = \frac{E_h}{E_b} = \frac{2E_h}{Q}$ $\quad s = \left(E^{e-} + E^{e+}\right)^2 = Q^2 > 0$



The production of a hadron h is described as two sequential events (factorization):
i) the production of the $q\bar{q}$ pair
ii) the fragmentation of the q or $\bar{q}$ into the hadron h

The summation runs over all quark flavors, because the detector cannot observe the quantum numbers of the parent of the detected hadron.
If we observe a meson containing a b quark (B – meson), very likely the meson came from the fragmentation of a b quark, because it is very unlikely to produce a b quark in the fragmentation process (very large mass!). We cannot say the same for a π meson.

The fragmentation function *D(z)* describes the transition parton → hadron in the same way that the structure function *f(x)* describes the embedding hadron → parton.
From the quantum field theory point of view, they are essentially the same objects.

Momentum and probability constraints impose

$$\sum_h \int_0^1 z \cdot D_q^h(z) \quad dz = 1$$

$$\sum_q \int_{z_{\min}}^1 \left[D_q^h(z) + D_{\bar{q}}^h(z) \right] \; dz = n_h$$

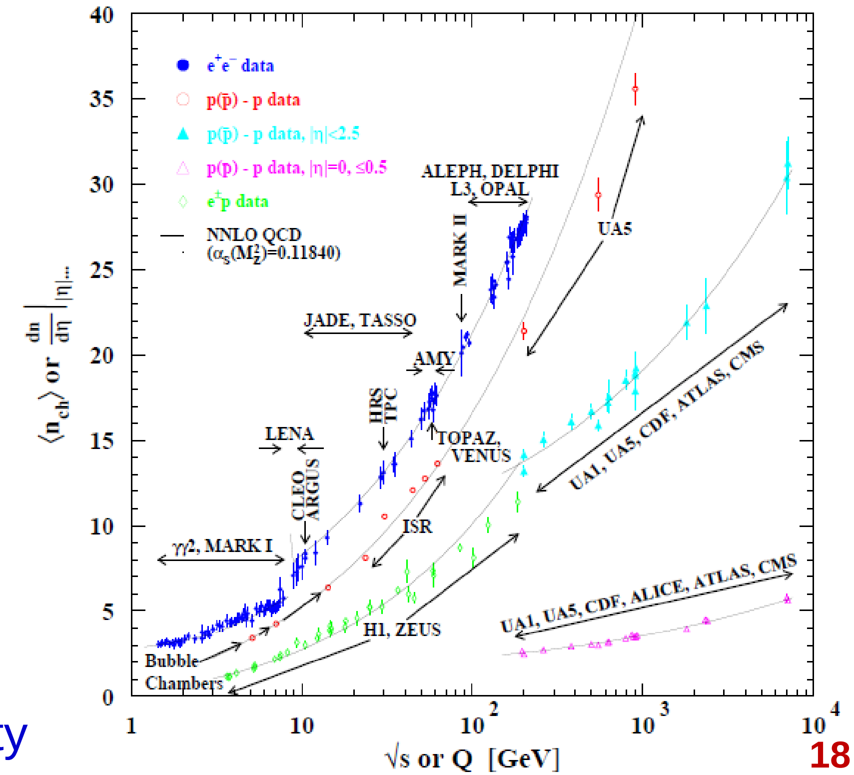
$z_{min} = 2m_h/Q$ = minimal energy to produce a hadron of mass m_h

n_h is the average multiplicity of hadrons h

The first equation simply states that the sum of energies of all hadrons is the energy of the initial quark.

It follows that the average hadron multiplicity n_h grows only logarithmically with *Q*:

$$n_h \sim \log\left(\frac{Q}{2m_h}\right)$$



average charged particle multiplicity

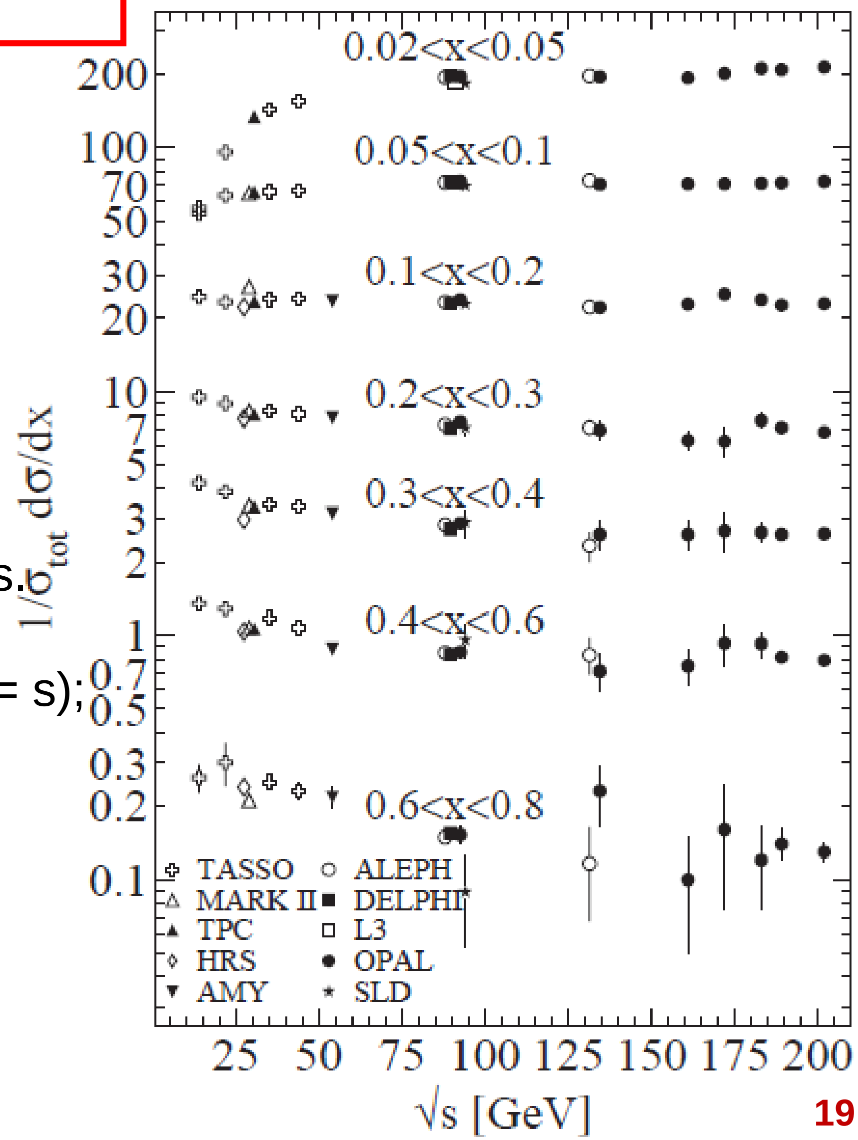
Dividing by the total annihilation cross section into hadrons

$$\frac{1}{\sigma^{e^+e^-\to had}}\frac{d\sigma\left(e^+e^-\to hX\right)}{dz}=\frac{\sum_q e_q^2\left[D_q^h(z)+D_{\bar{q}}^h(z)\right]}{\sum_q e_q^2}$$

1/σ dσ/dz is predicted to scale
(no Q dependence).

No surprise, since we based our derivation on the scaling properties of the parton model to introduce the fragmentation functions.

Data however show that the scaling is not perfect as is the case for the structure functions. Gluon emission from q and/or $\bar{q}$ will introduce logarithmic Q^2 scaling violations ($Q^2 = s$); their qualitative trend is similar to the one observed in electro-production (DIS): the density will increase at small z and decrease at large z.

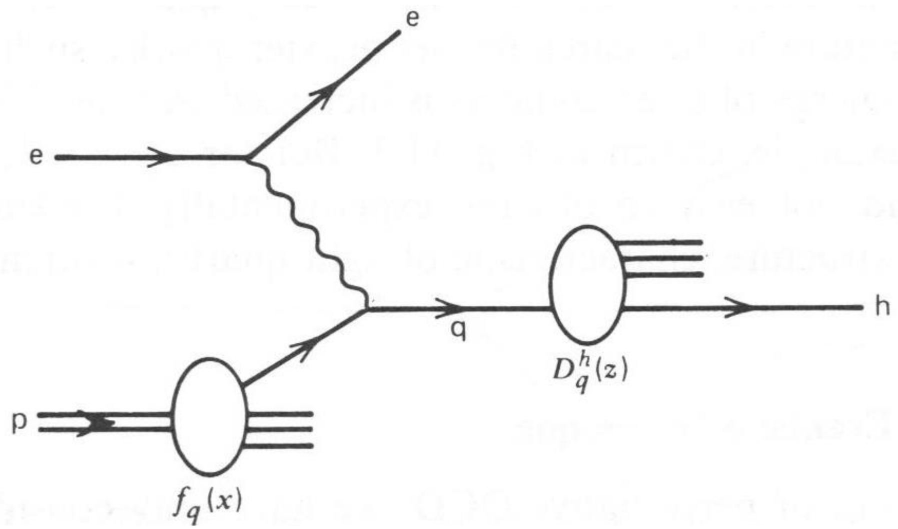


QCD cannot describe the shape of $D(z,Q_0^2)$, however it can describe its Q^2 evolution.

Fragmentation in DIS

The fragmentation functions D(z) describes universal properties of partons, like the structure functions f(x), no matter how the partons were produced.
We can analyze the production of hadrons in lepton induced processes (DIS) in a similar way, the result will be very similar

$$\frac{1}{\sigma^{ep\to e'X}}\frac{d\sigma\left(ep\to e'hX\right)}{dz}=\frac{\sum_q e_q^2 f_q(x)D_q^h(z)}{\sum_q e_q^2 f_q(x)}$$



where $f_q(x)$ are the parton density distribution functions.

Using charge conjugation and isospin invariance we can show that
(consider the quark content of the meson: if the quark is a valence quark like u in π^+, the fragmentation to π^+ is favored, otherwise we must produce two quarks in the fragmentation process and the fragmentation is unfavored)

favored fragmentation

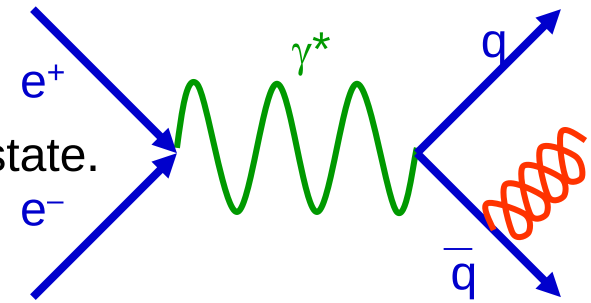
$$D_u^{\pi+}=D_{\bar{u}}^{\pi-}=D_d^{\pi-}=D_{\bar{d}}^{\pi+}$$

unfavored fragmentation

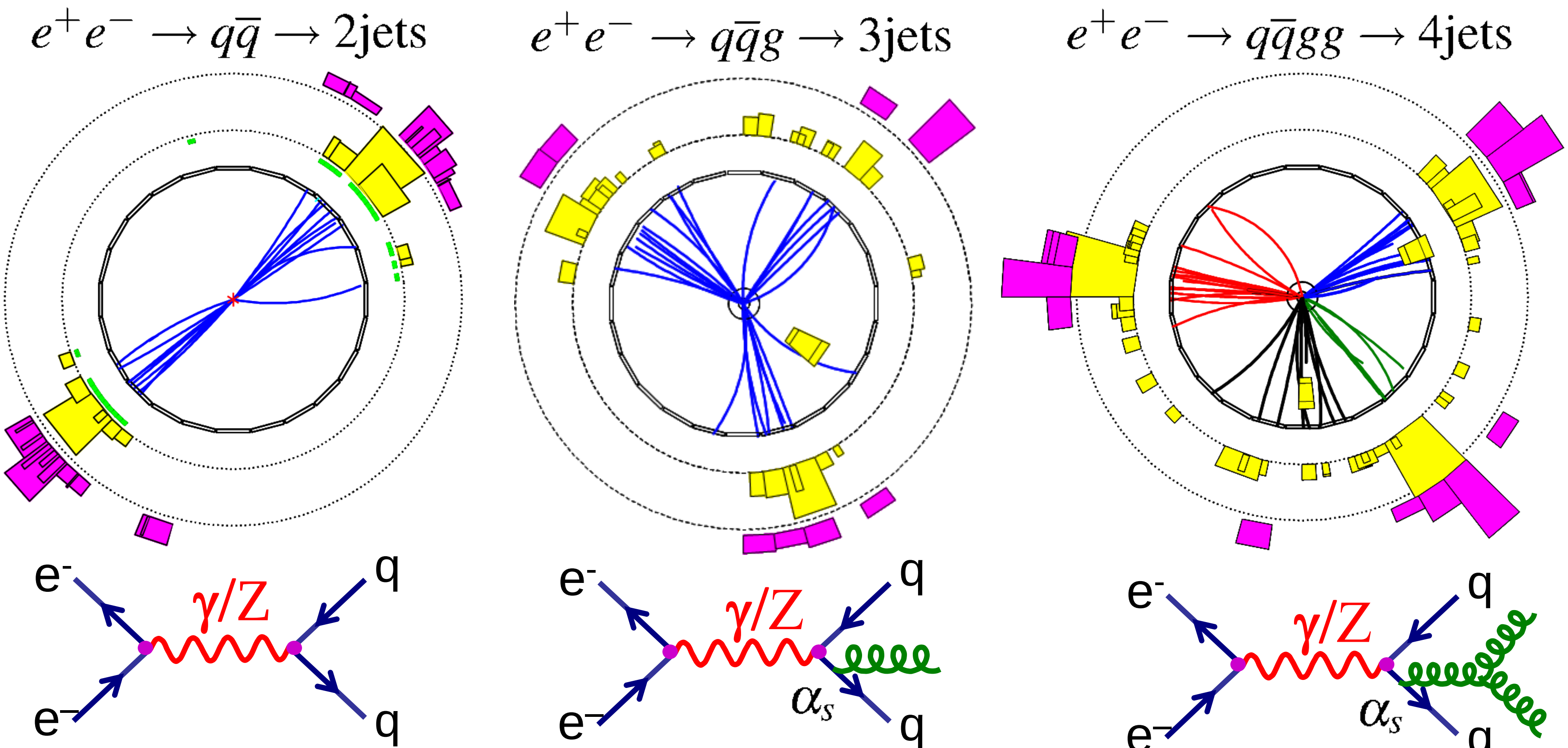
$$D_u^{\pi-}=D_{\bar{u}}^{\pi+}=D_d^{\pi+}=D_{\bar{d}}^{\pi-}$$

$$D_s^{\pi+}=D_s^{\pi-}$$

$e^+ e^- \to q \bar{q} g$



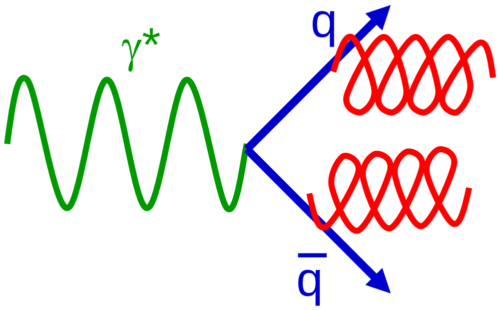
At order $\alpha^2\alpha_S$, the q or $\bar{q}$ can emit a gluon:
$e^+ e^- \to q \bar{q} g$ events are characterized by 3 jets in the final state.
The additional jet coms from gluon fragmentation.



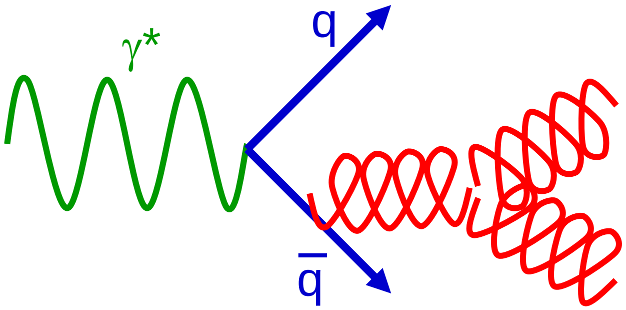
Three jet rate (~10% of events) ⇒ measurement of α_S
Angular distributions ⇒ gluons are spin-1 particles
Four-jet rate and distributions ⇒ triple gluon vertex (test underlying SU(3) symmetry)

4-jet Events

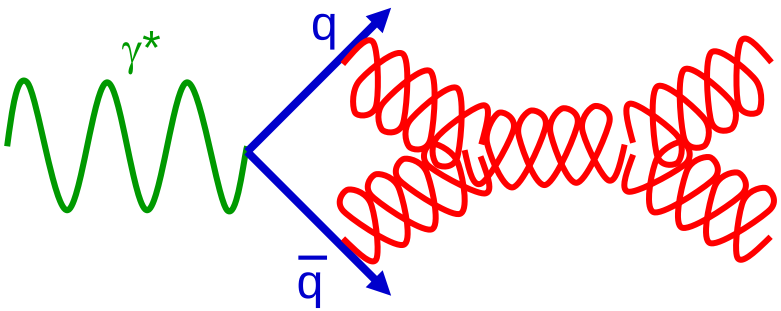
g_s^2 diagrams



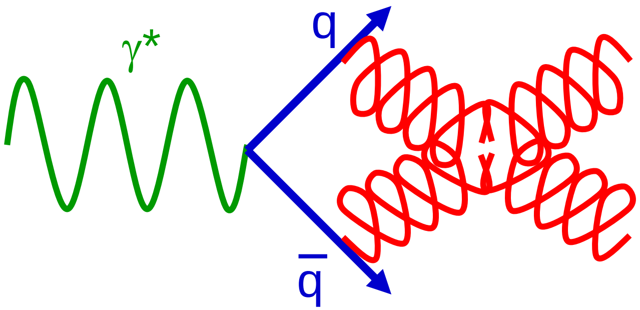
but also



g_s^4 diagrams



and

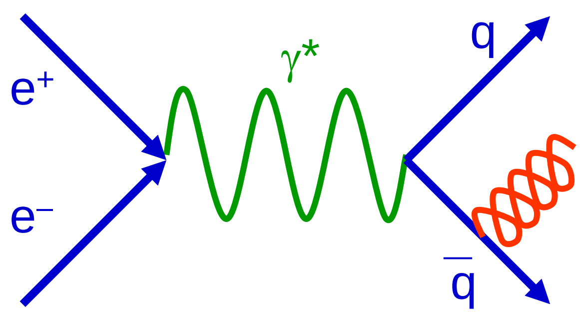


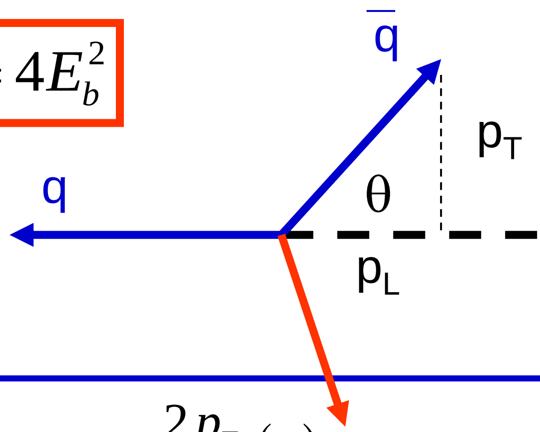
The diagrams on the right are allowed only in a non-abelian theory.
From the study of angular correlations in 4-jet events it has been shown that these diagrams are required to describe the data.

Kinematics of q $\bar{q}$ g

We work in the center of mass of the e⁺e⁻ system, i.e. the rest frame of the γ^*, and introduce variables normalized to the beam energy:

$$s = Q^2 = 4E_b^2$$





all 3 jets are in the same plane → coplanar

$$x_q \equiv \frac{2E_q}{Q} \qquad x_{\bar{q}} \equiv \frac{2E_{\bar{q}}}{Q} \qquad x_g \equiv \frac{2E_g}{Q} \qquad x_T \equiv \frac{2p_T}{Q}(\bar{q}) \qquad 0 \leq x \leq 1$$

The most obvious experimental signature of gluon emission is that the q and $\bar{q}$ jets are no longer produced back to back:

the $\bar{q}$ is produced with a transverse momentum x_T relative to the direction of q .

The three jets, however, are coplanar.

The four momentum *fractions* are (we assume the quarks are massless)

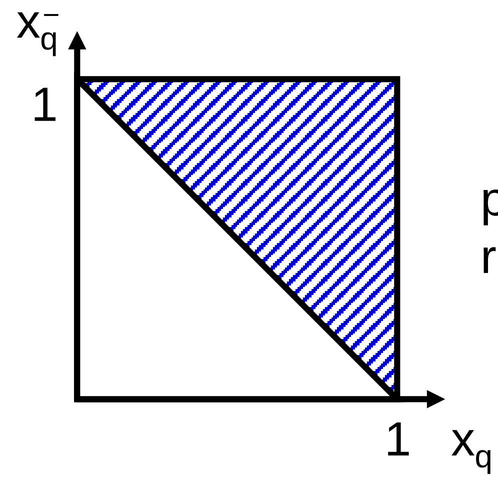
for the quark $(x_q, 0, 0, x_q)$

for the antiquark $(x_{\bar{q}}, x_T, 0, -x_L)$

for the gluon $(x_g, -x_T, 0, -x_q + x_L)$

Energy conservation imposes

$$x_q + x_{\bar{q}} + x_g = 2 \quad \Rightarrow x_q + x_{\bar{q}} = 2 - x_g \geq 1$$



For massles quarks and gluons (4-momentum squared)

$$x_{\bar{q}}^2 - x_T^2 - x_L^2 = m_{\bar{q}}^2 = 0$$
$$x_g^2 - x_T^2 - \left(x_L - x_q\right)^2 = m_g^2 = 0$$

and express the fractional transverse momentum x_T in terms of x_q, x_g, …

$$x_T^2 = \frac{4}{x_q^2}\left(1 - x_q\right)\left(1 - x_{\bar{q}}\right)\left(1 - x_g\right)$$

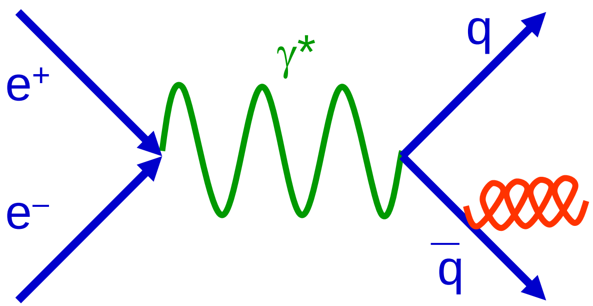
$$x_{\bar{q}} = \frac{2\left(1 - x_q\right)}{2 - x_q - x_q \cos\theta}$$

Experimentally one can recognize the quarks and gluon jets by ordering the jets in energy

$$E_1 \geq E_2 \geq E_3$$

with a good probability that the most energetic jet is the one that did not emit the gluon and that the least energetic jet is the one initiated by the gluon emitted by the second jet. Because of different color factors involved in the fragmentation of quarks and gluons, the gluon jet can also be broader with higher hadron multiplicities.

The Cross Section



In this graph the $\bar{q}$ emits a softer gluon, hence

$$x_q \geq x_{\bar{q}} \geq x_g$$

Since the most evident signature for gluon emission is $x_T = 2p_T / Q$ the relevant observable will be $d\sigma / dx_T^2$ w.r.t. the axis defined by *q*.

The cross section can be calculated in two steps:

1. γ^* flux at the $e^+e^- \to \gamma^*$ vertex

2. $\gamma^* \to$ q q g diagram

Using the Altarelli-Parisi technique we can reuse the formalism developed in DIS

$$\frac{d\sigma}{dx_{\bar{q}}\,dp_T^2} = \sigma\left(e^+e^- \to q\bar{q}\right)\ \gamma_{\bar{q}\leftarrow\bar{q}}\left(x_{\bar{q}}, p_T^2\right)$$

where σ is the cross section for producing a $q\,\bar{q}$ pair and γ_{qq} is the probability that the $\bar{q}$ emits a gluon with momentum $1\text{-}x_{\bar{q}}$ and transverse momentum p_T

$$\sigma\left(e^+e^- \to q\bar{q}\right) = \frac{4\pi\alpha^2}{Q^2} e_q^2$$

$$\gamma_{\bar{q}\leftarrow\bar{q}}\left(x_{\bar{q}}, p_T^2\right) = \gamma_{q\leftarrow q}\left(x_q, p_T^2\right) = \frac{\alpha_S}{2\pi}\frac{1}{x_T^2} P_{q\leftarrow q}(x_{\bar{q}})$$

with the splitting function

$$P_{q\leftarrow q}(x) = \frac{4}{3}\left(\frac{1+x^2}{1-x}\right)$$

Inserting γ_{qq} and dividing by σ we obtain

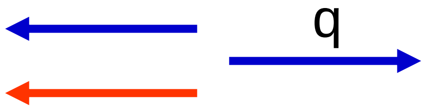
$$\frac{1}{\sigma}\frac{d\sigma}{dx_{\bar{q}}dx_T^2} = \frac{\alpha_S}{2\pi}\frac{1}{x_T^2}P_{q\leftarrow q}(x_{\bar{q}})$$

Integrating over all possible $\bar{q}$ energies $dx_{\bar{q}}$ we obtain
(a factor of 2 is inserted to account for the case when the q radiates the gluon)

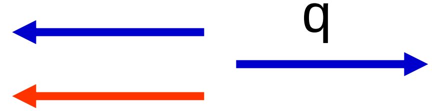
$$\frac{1}{\sigma}\frac{d\sigma}{dx_T^2} = 2\frac{\alpha_S}{2\pi}\frac{1}{x_T^2}\int_{(x_{\bar{q}})min}^{(x_{\bar{q}})max} dx \frac{4}{3}\left(\frac{1+x^2}{1-x}\right)$$

This cross section diverges for $x_q \rightarrow 1$ or $x_{\bar{q}} \rightarrow 1$. These are the so called collinear divergences and appear often in first order calculations.

$x_{\bar{q}} = 1$

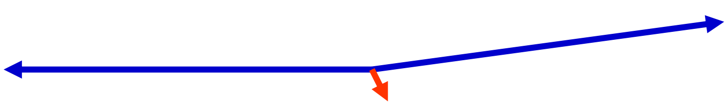


$x_q = 1$



Experimentally we cannot distinguish a q ($\bar{q}$) jet superposed to a gluon jet $\rightarrow$ collinear divergence or a q accompanied by a soft (very low energy) gluon $\rightarrow$ infrared divergence (i.e. energy below the detection threshold).

In the limit $x_{\bar{q}} \rightarrow 1$, $x_{\bar{q}} = x_q$:
this can happen if the emitted gluon is as "soft" as possible $\Rightarrow x_g \approx x_T$



$x_g \sim x_T$

$$\left(x_q\right)_{min} = \left(x_{\bar{q}}\right)_{\max} \approx 1 - \frac{x_T}{2}$$

$$x_g = x_T \quad \text{for} \quad \theta_g \rightarrow 0$$

These divergencies however are not a serious problem as long as we compare measurements to predictions: we integrate over the "experimental" phase space and not over the whole phase space:
we require three reconstructed jets, i.e. cannot see two overlapping jets ($\to x_{\bar{q}} < 1$) and we have limited resolution to see if the two jets are back-to-back or not ($\to x_q < 1$).

Finally we can perform our integral
(recall the cut-offs introduced in the derivation of the Altarelli-Parisi equations:
$p_{T\,min}$ and $p_{T\,max}$, max = maximal available energy)

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}x_T^2} \approx \frac{4}{3}\frac{\alpha_s}{\pi}\frac{1}{x_T^2}\int_{(x_{\bar{q}})_{\min}}^{1-\frac{1}{2}x_T}\frac{2\mathrm{d}x}{1-x}$$

where we have approximated $(1 + x^2)$ by 2.

$$\int_{(x_{\bar{q}})_{\min}}^{1-\frac{1}{2}x_T}\frac{\mathrm{d}x}{1-x} = \log(1-x)\Big|_{1-\frac{1}{2}x_T}^{(x_{\bar{q}})_{\min}} = \log(1-0)-\log(x_T/2) = \log(1/x_T)+\cancel{\log(2)} \to \frac{1}{2}\log(1/x_T^2)$$

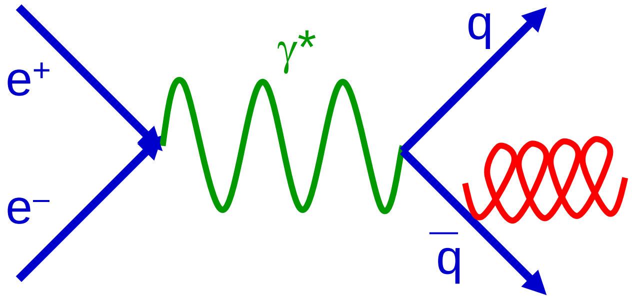
Keeping only the leading logarithmic term, we finally obtain

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}x_T^2} \approx \frac{4\alpha_s}{3\pi}\frac{1}{x_T^2}\log\left(\frac{1}{x_T^2}\right) \quad \text{or} \quad \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}p_T^2} \sim \frac{4\alpha_s}{3\pi}\frac{1}{p_T^2}\log\left(\frac{Q^2}{4p_T^2}\right)$$

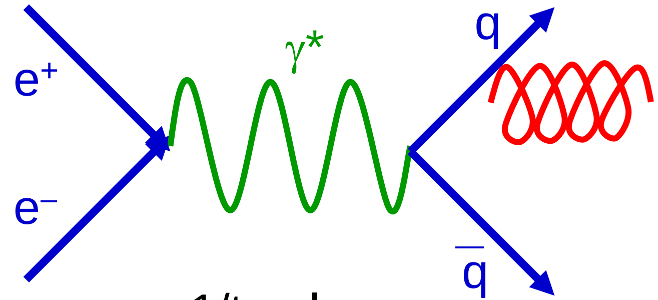
The cross section increases with energy ($s = Q^2$) for fixed p_T. This results from the increased probability of emitting a gluon when the annihilation energy increases.

The Cross Section from Feynman Diagrams

We have to consider two diagrams (gluon radiated by a q and gluon radiated by a $\bar{q}$)



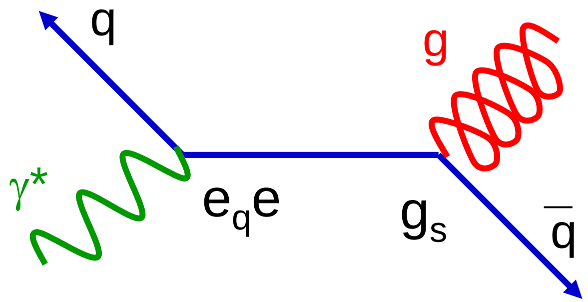
1/s pole



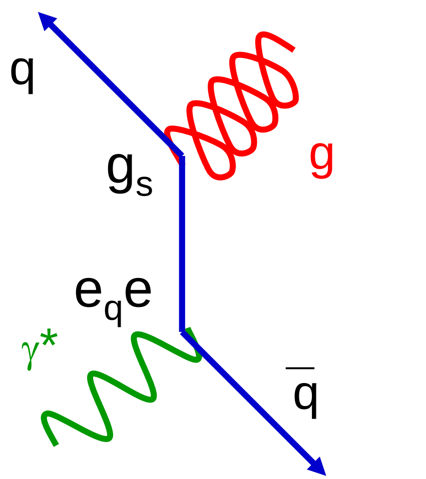
1/t pole

The diagram $\gamma^* q \to qg$ ($\gamma^* \bar{q} \to \bar{q} g$) is analogue to the Compton scattering $\gamma^* q \to \gamma q$ (bremsstrahlung with a γ^* !), except that for the gluon vertex we have to use α_S and the color factors instead of α.

We have to add these two diagrams



+



Proceeding in the same way as before (L4) we can calculate the <|amplitude|^2> (note the sign of the first two terms).

$$\left\langle |M|^2 \right\rangle = 32\pi^2 \left(e_q^2 \alpha \alpha_S \right) \frac{4}{3} \left(\frac{\hat{t}}{\hat{s}} + \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u} Q^2}{\hat{s}\hat{t}} \right)$$

Express the Mandelstam variables s, t, u in terms of energy fraction variables x_q, x_g, …:

$$\hat{s} = \left(p_{\gamma*} - p_q\right)^2 = Q^2 - 2QE_q = Q^2(1 - x_q) > 0$$

$$\hat{t} = \left(p_{\gamma*} - p_{\bar{q}}\right)^2 = Q^2 - 2QE_{\bar{q}} = Q^2(1 - x_{\bar{q}}) > 0 \qquad p_{\gamma*}^2 = Q^2$$

$$\hat{u} = \left(p_{\gamma*} - p_g\right)^2 = Q^2 - 2QE_g = Q^2(1 - x_g) > 0$$

The <|amplitude|^2> becomes

$$\left\langle |M|^2 \right\rangle = 32\pi^2 \left(e_q^2 \alpha \alpha_S\right) \frac{4}{3} \left(\frac{\hat{t}}{\hat{s}} + \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}} \right)$$

$$= 32\pi^2 \left(e_q^2 \alpha \alpha_S\right) \frac{4}{3} \frac{Q^4(1 - x_{\bar{q}})^2 + Q^4(1 - x_q)^2 + 2Q^4(1 - x_g)^2}{Q^4(1 - x_q)(1 - x_{\bar{q}})}$$

$$= 32\pi^2 \left(e_q^2 \alpha \alpha_S\right) \frac{4}{3} \frac{x_{\bar{q}}^2 + x_q^2}{(1 - x_q)(1 - x_{\bar{q}})}$$

and the exact O(α_S) result is (after dividing by σ)

$$\frac{1}{\sigma} \frac{d\sigma}{dx_q dx_{\bar{q}}} = \frac{\alpha_S}{2\pi} \frac{4}{3} \frac{x_q^2 + x_{\bar{q}}^2}{(1 - x_q)(1 - x_{\bar{q}})}$$

which is diverges for $x_q \to 1$ as before. Note the symmetry between q and $\bar{q}$.
The previous result is the so called leading logarithmic approximation.

To complete the discussion let's compare the two results:
we have to transform x_q to x_T^2

$$\frac{d\sigma}{dx_{\bar{q}}dx_T^2} = \frac{d\sigma}{dx_{\bar{q}}dx_q}\frac{dx_q}{dx_T^2}$$

In the small p_T approximation we have

$$\left|\frac{dx_T^2}{dx_q}\right| \approx 4x_{\bar{q}}(1-x_{\bar{q}})$$

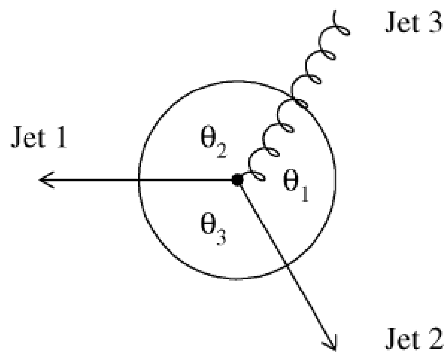
for $x_q \sim 1$ (in this limit $x_{\bar{q}} \sim (1 - x_g)$)

which allow us to rewrite the cross section as

$$\frac{d\sigma}{dx_{\bar{q}}dx_T^2} \approx \frac{8}{3}\frac{\alpha^2\alpha_S}{Q^2}e_q^2\left(\frac{1+x_{\bar{q}}^2}{1-x_{\bar{q}}}\right)\underbrace{\left[\frac{1}{4(1-x_q)(1-x_{\bar{q}})x_{\bar{q}}}\right]}_{= 1 / x_T^2}$$

(cfr. slide 25)

Spin of the Gluon

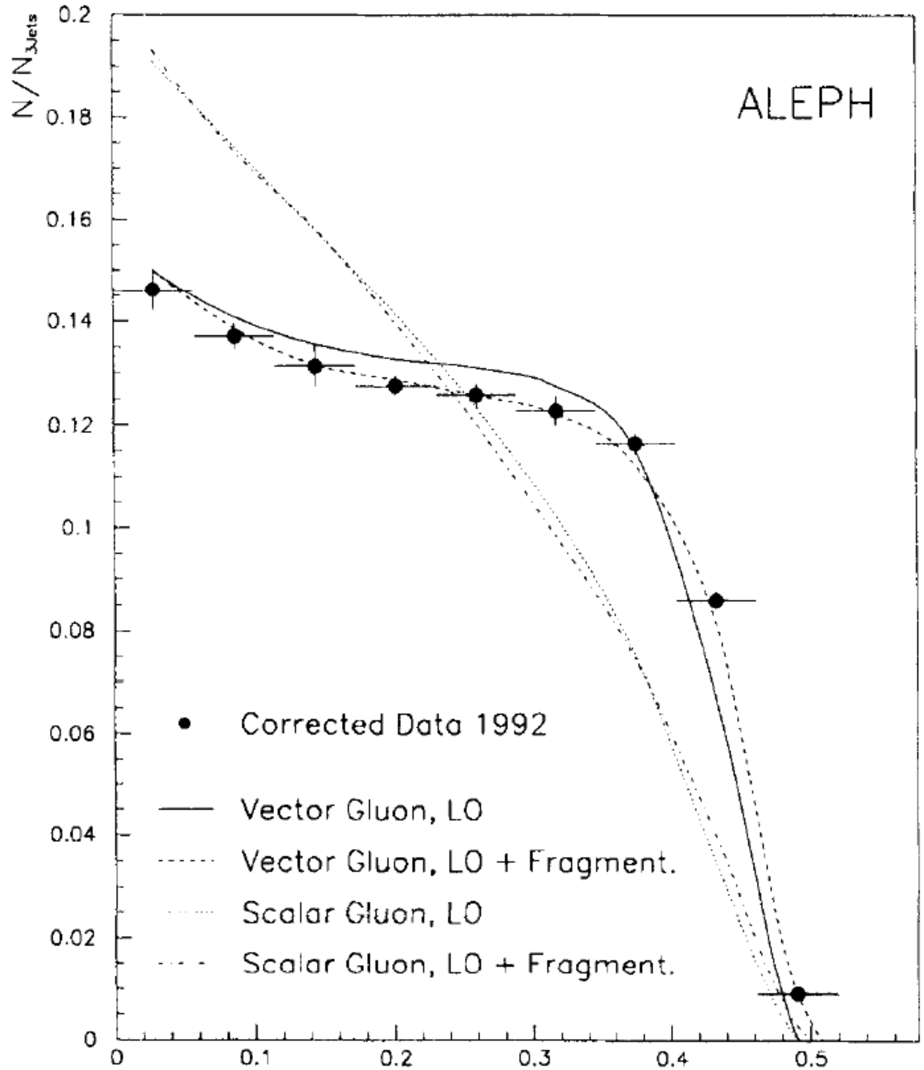


If the gluon had spin 0, the form of the cross section would be quite different. In this derivation we assumed that the q – g coupling is a vector coupling.

To try to *identify* the gluon jet we order the jets in energy

$$E_1 > E_2 > E_3$$

There is a good chance that E_3 is the radiated gluon, E_1 is the (anti) quark jet that did not radiate the gluon, and E_2 the quark jet that radiated the gluon.



x_g

data indicate that
gluons have spin 1

Dealing With the Divergences

To calculate the QCD corrections to *R*, we must integrate the cross sections over both x_q an $x_{\bar{q}}$ from 0 to 1. We encounter the common problem of divergences for $x_q \to 1$ or $x_{\bar{q}} \to 1$.
Let's have a closer look:

$$1 - x_q = \frac{\hat{s}}{Q^2} = \frac{(p_{\bar{q}} + p_g)^2}{Q^2} \approx \frac{2 p_{\bar{q}} \cdot p_g}{Q^2} = \frac{2}{Q^2} E_{\bar{q}} E_g \left(1 - \cos\theta_{\bar{q}g}\right)$$

soft gluon – infrared divergence

collinear divergence

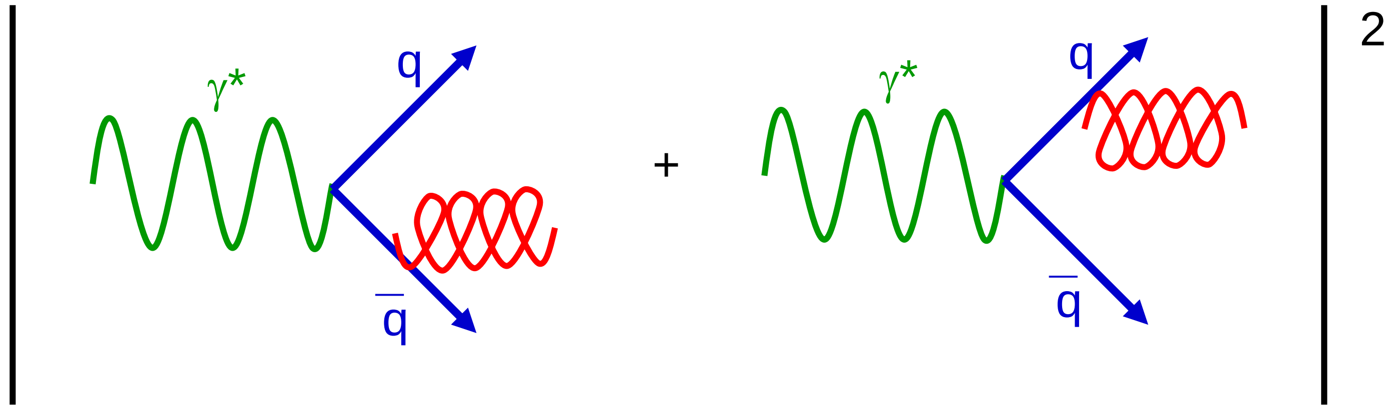
(1 - x_q) vanishes when the gluon becomes very soft, i.e. $E_g \to 0$ (infrared divergence) or when the $\bar{q}$ and g become collinear (collinear divergence or mass singularity: if the quark or gluon had mass, cos θ = 1 would be kinematically impossible).

To regularize these divergences (note this is not renormalization, renormalization deals with ultraviolet divergences) we give a fictitious mass m_g to the gluon, i.e. repeat the calculations of the Feynman diagrams with $m_g \neq 0$.

To be completed, the calculations must include all contributions of the same order in α_S, i.e. we must include also the virtual gluon diagrams (i.e. loops).

A lengthy calculation (kinematical factors!) for real gluon emission, σ_{real} , gives

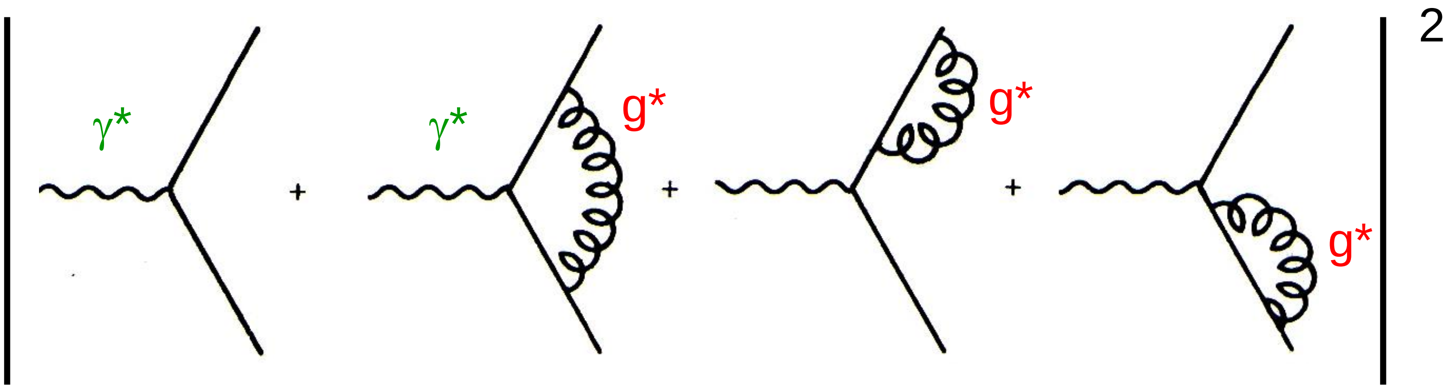
σ_{real} =



$$\sigma_{real} = \int dx_q dx_{\bar{q}} \frac{d\sigma}{dx_q dx_{\bar{q}}} = \sigma\left(e^+e^- \to q\bar{q}\right)\frac{\alpha_S}{2\pi}\frac{4}{3}\left\{\log^2\left(\frac{m_g}{Q}\right) + 3\log\left(\frac{m_g}{Q}\right) - \frac{\pi^2}{3} + 5\right\}$$

As anticipated, it is divergent for $m_g \to 0$. However this cannot be the final answer, since the result cannot depend on m_g. To be complete, the calculation at the same order in α_S must include also the virtual gluon diagrams:

$\sigma_{virtual}$ =



The interference between the first diagram and the virtual gluon loops (last three) leads to an additional term of order α_S ! that we will label $\sigma_{virtual}$.

Again, after a lengthy calculation, one can arrive at

$$\sigma_{virtual} = \sigma\left(e^+e^- \to q\bar{q}\right)\frac{\alpha_S}{2\pi}\frac{4}{3}\left\{-\log^2\left(\frac{m_g}{Q}\right) - 3\log\left(\frac{m_g}{Q}\right) + \frac{\pi^2}{3} - \frac{7}{2}\right\}$$

Surprisingly enough, the log terms are identical to the ones that we encountered when calculating σ_{real}, but with opposite sign.

The total α_S contribution is the sum of both, σ_{real} and $\sigma_{virtual}$:

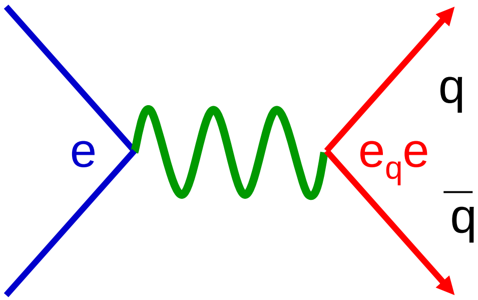
$$\sigma(\alpha_S) = \sigma_{real} + \sigma_{virtual} = \sigma\left(e^+e^- \to q\bar{q}\right)\frac{\alpha_S}{2\pi}\frac{4}{3}\left\{5 - \frac{7}{2}\right\} = \sigma\left(e^+e^- \to q\bar{q}\right)\frac{\alpha_S}{\pi}$$

The cancellation of the singularities between the contributions with the emission of real gluons and virtual gluons occurs in several processes, for instance in deep inelastic scattering. We encounter it also in QED.
The cancellation between real and virtual photon (gluon) emission occurs at all orders in the perturbative expansion.

Including the α_S corrections to *R* we finally obtain

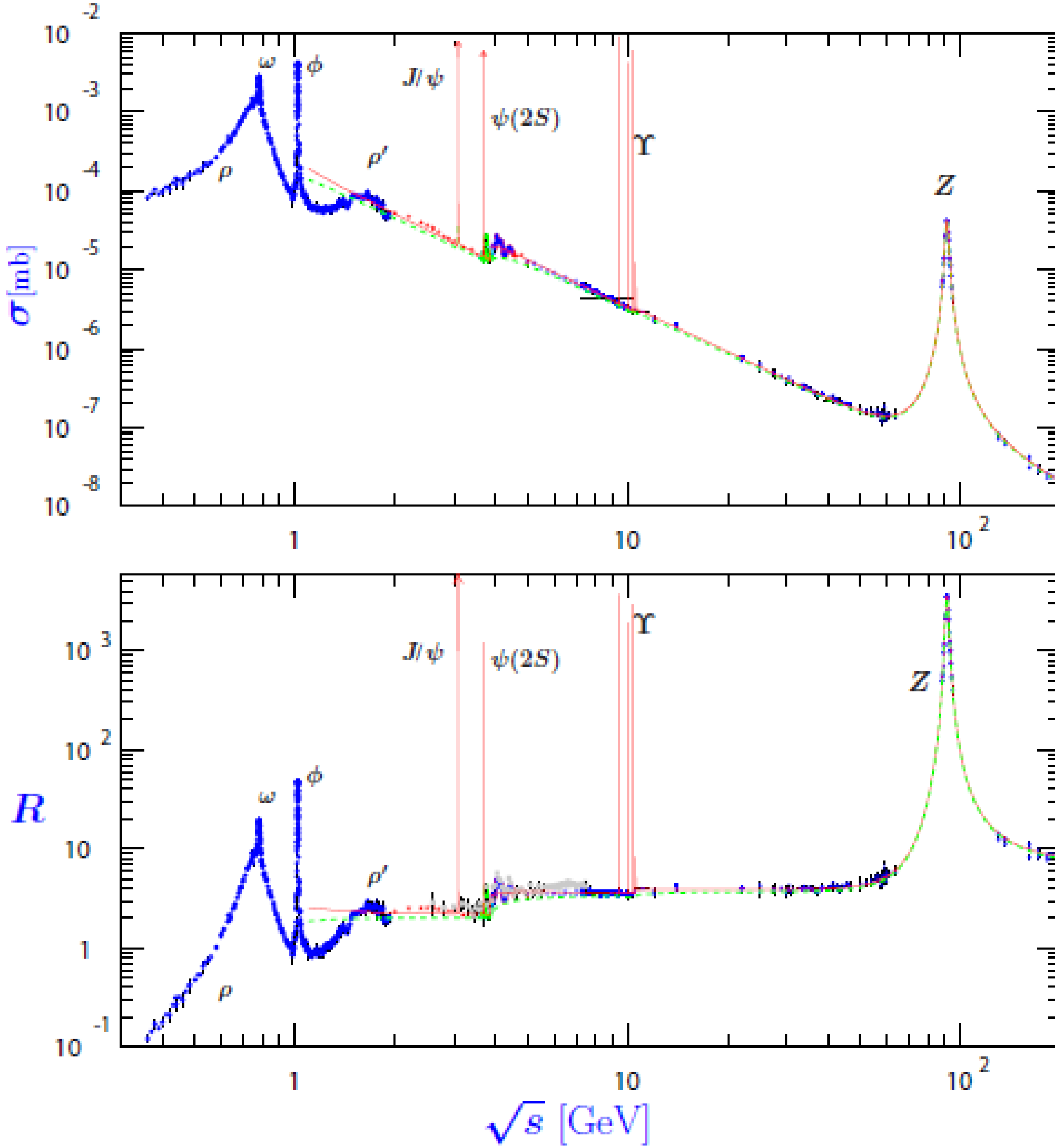
$$R = \frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\sum_q e_q^2\left(1 + \frac{\alpha_S}{\pi} + 1.41\left(\frac{\alpha_S}{\pi}\right)^2 - 12.8\left(\frac{\alpha_S}{\pi}\right)^3 + \ldots\right)$$

The Ratio R



in green
the QPM prediction

in red
with $\alpha_S{}^3$ corrections



For Next Week

Study the material and prepare / ask questions
Study ch. 11 (sec. 1 to 7) and / or ch. 10 (sec. 6) in Thomson

Do the homeworks

Next week we will study the hadron – hadron interactions
 have a first look at the lecture notes, you can already have questions
 read ch. 11 (sec. 8, 9) in Halzen & Martin and / or ch. 10 (sec. 9) in Thomson