Advanced Particle Physics 2

Strong Interactions and Weak Interactions

L11 – Electro-Weak Interactions

(http://dpnc.unige.ch/~bravar/PPA2/L11)

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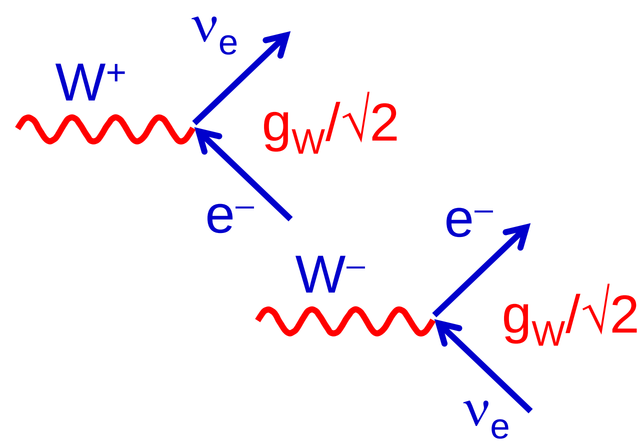
Electro-Weak Interactions

weak interaction phenomenology
$\rightarrow$ put on solid grounds:

- identify the underlying symmetries
- $\rightarrow$ gauge invariance (massive gauge bosons?)
- renormalizability

charge raising $J_\mu^{CC}(+) = \bar{\nu}_L \gamma_\mu e_L$



charge lowering $J_\mu^{CC}(-) = \bar{e}_L \gamma_\mu \nu_L$

Introduce $\chi_L = \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L$ left-handed doublet and (weak isospin doublet) (ν_R) e_R^- right-handed singlet(s)

and rewrite the charged currents as

charge raising $J_\mu^{CC}(+) = \bar{\chi}_L \gamma_\mu \tau_+ \chi_L$

charge lowering $J_\mu^{CC}(-) = \bar{\chi}_L \gamma_\mu \tau_- \chi_L$

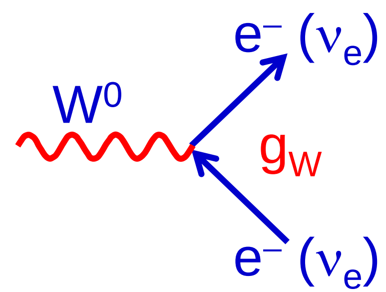
$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\tau_+^\dagger = \tau_- \qquad \tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$$

τ_i – Pauli matrices

The Weak Current Triplet

Add the 3rd component to the J_μ^+ and J_μ^- currents, i.e.



$$J_\mu^3 = \bar{\chi}_L \gamma_\mu \frac{1}{2}\tau_3 \chi_L = \frac{1}{2}\bar{\nu}_L \gamma_\mu \nu_L - \frac{1}{2}\bar{e}_L \gamma_\mu e_L$$

[it cannot be identified with the γ or J_μ^{EM} (massless, also a right-handed component!).}

and introduce the "weak isospin" triplet of weak currents

$$J_\mu^i = \bar{\chi}_L \gamma_\mu \frac{1}{2}\tau_i \chi_L \qquad \rightarrow \qquad T_i = \int d^3x \, J_0^i(x)$$

$$\left[T_i, T_j\right] = i\varepsilon_{ijk} T_k$$

by construction, τ_i matrices

The "charges" T_i generate an $SU(2)_L$ algebra of left-handed weak currents.

Note: $J_\mu^+ = 1/2(J_\mu^1 + i\, J_\mu^2)$ and $J_\mu^- = 1/2(J_\mu^1 - i\, J_\mu^2)$

Can also introduce the triplet of weak vector bosons

$$\begin{pmatrix} W^1 \\ W^2 \\ W^3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2}(W^+ + W^-) \\ -i/\sqrt{2}(W^+ - W^-) \\ W^0 \end{pmatrix} \qquad \begin{pmatrix} W^+ \\ W^0 \\ W^- \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2}(W^1 + iW^2) \\ W^3 \\ 1/\sqrt{2}(W^1 - iW^2) \end{pmatrix}$$

The Electro Magnetic Interaction

Let's rewrite the electromagnetic current as (here e represents the electron spinor)

$$J_\mu^{EM} = \bar{e}\gamma_\mu Q e = \bar{e}_R \gamma_\mu Q\, e_R + \bar{e}_L \gamma_\mu Q\, e_L \qquad \rightarrow \qquad Q = \int d^3x\, J_0^{EM}(x)$$

with Q the charge operator (with eigenvalue -1 for the electron).
Q is the generator of the $U(1)_{EM}$ symmetry group.

J_μ^{EM} contains left-handed and right-handed components with equal weights.

We could stop here, however J_μ^3 and W^0 have never been observed.
J_μ^{EM} does not belong to $SU(2)_L$ since it contains left-handed and right-handed components.

In the attempt to save the $SU(2)_L$ symmetry include also J_μ^{EM} (Glashow 1961) and enlarge the symmetry group $\rightarrow$ electro-weak unification.
By combining J_μ^3 and J_μ^{EM} with different weights, one could build the physical current J_μ^{NC}, which contains left- and right-handed components. This, however, will require the introduction of a new weak hypercharge current J_μ^Y orthogonal to J_μ^3.

"Unification"

Both "neutral" currents J_μ^{NC} and J_μ^{EM} contain left-handed and right-handed components. Neither respects the $SU(2)_L$ symmetry. Nan can be identified with J_μ^3.

Glashow's proposal (1961), well before the discovery of Neutral Currents (1973):

Form two orthogonal combinations starting with J_μ^{NC} and J_μ^{EM} .
These two new currents must have definite transformation properties under $SU(2)_L$:

one combination, J_μ^3, with coupling g is to complete the weak isospin triplet J_μ^i and it is purely left-handed;

the second combination, orthogonal to J_μ^3, is the new weak hypercharge current J_μ^Y introduced by Glashow with coupling g'/2. J_μ^Y is a singlet under $SU(2)_L$.

The weak hypercharge current J_μ^Y contains right-handed and left-handed components (although with different weights):

$$J_\mu^Y = \bar{\psi}\,\gamma_\mu Y \psi \qquad \rightarrow \qquad Y = \int \mathrm{d}^3x\; J_0^Y(x)$$

The weak hypercharge operator Y is defined by

$$Q = T_3 + \frac{1}{2}Y \quad \rightarrow \quad Y = 2\left(Q - T_3\right)$$

Y generates the $U(1)_Y$ symmetry group with B_μ^0 the associated gauge boson.

We have two "neutral currents", J_μ^3 and J_μ^Y (associated to the W^0 and B^0), nan physical.
The photon J_μ^{EM} is a linear superposition of the J_μ^3 and J_μ^Y currents with equal amounts of left- and right-handed components.
A new current, J_μ^{NC}, orthogonal to J_μ^{EM}, is thus predicted, with different amounts of left- and right-handed components.

All this might work, if e ~ g' (i.e. similar strength).

The electromagnetic current In terms of J_μ^3 and J_μ^Y reads

$$J_\mu^{EM} = J_\mu^3 + \frac{1}{2} J_\mu^Y$$

The weak hypercharge current for the electron (Q = -e) is given by

$$\begin{aligned} J_\mu^Y &= 2J_\mu^{EM} - 2J_\mu^3 \\ &= -2(\bar{e}_R \gamma_\mu e_R + \bar{e}_L \gamma_\mu e_L) - (\bar{\nu}_L \gamma_\mu \nu_L - \bar{e}_L \gamma_\mu e_L) \\ &= -2\left(\bar{e}_R \gamma_\mu e_R\right) - 1\left(\bar{\chi}_L \gamma_\mu \chi_L\right) \end{aligned}$$

We have incorporated the electromagnetic interaction and the symmetry group has been enlarged to $SU(2)_L \times U(1)_Y$.
In a sense we have unified the electromagnetic and weak interactions.
However two open issues remain:

- massive gauge bosons W and Z (Weinberg 1967 and Salam 1968 via Higgs mech.)
- renormalizability (t'Hooft 1975 using the Higgs field)

$$SU(2)_L \times U(1)_Y$$

Two symmetry groups and two coupling constants g_W (W) and $g_Y'/2$ (Y).

Classify all particles according to the weak isospin T and the hypercharge Y = $2(Q-T_3)$.

$$\chi_L^{leptons} = \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \quad (\nu_R) \quad e_R^- \qquad \chi_L^{quarks} = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad u_R \quad d_R$$

Note that $[Y,T_i] = 0$ (i.e. they commute → different symmetry groups).
⇒ all members of a weak isospin multiplet have the same weak hypercharge Y.

Weak Isospin and Weak Hypercharge quantum numbers for leptons and quarks

	T	T_3	Q	Y
$\nu_{e\,L}$	$1/2$	$1/2$	0	-1
e_L	$1/2$	$-1/2$	-1	-1
$(\nu_{e\,R})$	0	0	0	0
e_R	0	0	-1	-2

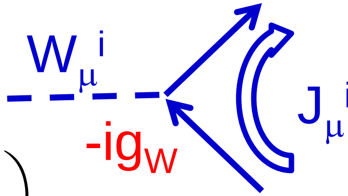
	T	T_3	Q	Y
u_L	$1/2$	$1/2$	$2/3$	$1/3$
d_L	$1/2$	$-1/2$	$-1/3$	$1/3$
u_R	0	0	$2/3$	$4/3$
d_R	0	0	$-1/3$	$-2/3$

u_R and d_R are singlets, i.e. they do not form a right-handed doublet.
Note that ν_R , if it exists, carries no quantum numbers,
i.e. it does not interact via any known force, not even the neutral current J_μ^{NC}.

The Basic Electro-Weak Interaction

Iso-triplet vector field, $W_\mu^i \rightarrow J_\mu^i$ coupling g_W
$SU(2)_L$ symmetry group

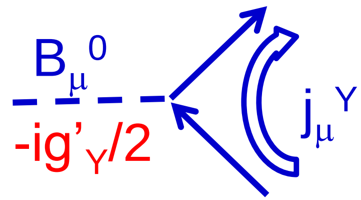
$$-ig_W \vec{J}_\mu \cdot \vec{W}^\mu = -ig_W \bar{\chi}_L \gamma_\mu \vec{T} \cdot \vec{W}^\mu \chi_L \quad \left(\vec{T} = \frac{1}{2}\vec{\tau} \right)$$



(there is no √2 because we are dealing with W^i)

Iso-singlet vector field, $B_\mu \rightarrow J_\mu^Y$ coupling $g_Y'/2$
$U(1)_Y$ symmetry group

$$-ig_Y' \frac{1}{2} J_\mu^Y B^\mu = -ig_Y' \bar{\psi} \gamma_\mu \frac{Y}{2} \psi B^\mu$$



Basic ElectroWeak interaction

$$-ig_W J_\mu^i W^{\mu,i} - ig_Y' \frac{1}{2} J_\mu^Y B^\mu = -i \frac{g_W}{\sqrt{2}} \left(J_\mu^+ W^{\mu,+} + J_\mu^- W^{\mu,-} + \sqrt{2} J_\mu^0 W^{\mu,0} \right) - ig_Y' \frac{1}{2} J_\mu^Y B^\mu$$

Weak Charged Bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \pm i W_\mu^2 \right)$$

Neutral Bosons

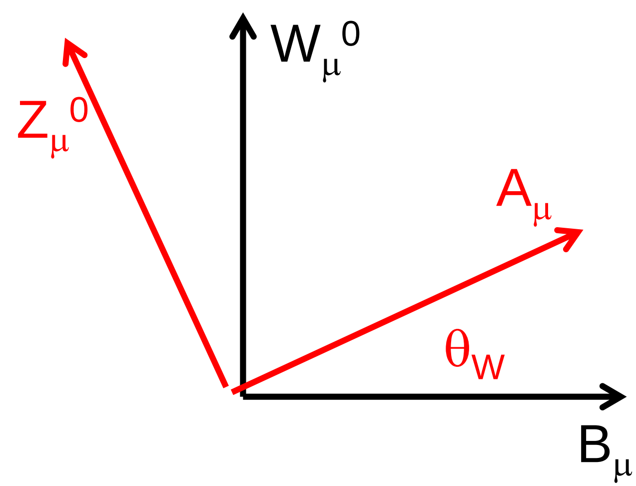
$$W_\mu^0 \quad \text{and} \quad B_\mu^0 \quad \rightarrow \quad Z^0 \quad \text{and} \quad \gamma$$

The Photon and the Z Boson

Express the "observed" massless photon field A_μ and the massive field Z_μ^0 in terms of W_μ^0 and B_μ^0

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\vartheta_W & \sin\vartheta_W \\ -\sin\vartheta_W & \cos\vartheta_W \end{pmatrix} \begin{pmatrix} B_\mu^0 \\ W_\mu^0 \end{pmatrix} \qquad \begin{aligned} A_\mu &= \cos\vartheta_W B_\mu^0 + \sin\vartheta_W W_\mu^0 \\ Z_\mu &= -\sin\vartheta_W B_\mu^0 + \cos\vartheta_W W_\mu^0 \end{aligned}$$

θ_W – electroweak mixing angle
(Weinberg angle, originally introduced by Glashow)



$$\sin^2\vartheta_W(M_Z) = 0.23126 \pm 0.00005$$

with the condition $$g\sin\vartheta_W = e = g'\cos\vartheta_W$$

Have we really unified the EM and Weak interactions?

We have started with two independent theories with couplings g_W and e and we have arrived at coupling constants which are related, but at the cost of introducing a new parameter, the Weinberg angle θ_W . The interactions are not unified from any "higher" principle, but it works! (For a "real" unification one would need a larger symmetry group containing $SU(2)_L$ and $U(1)_Y$ as subgroups with only one coupling constant.)

Electro-Weak Neutral Current Interaction

Express the electro-weak Neutral Current interaction in terms of the fields A_μ and Z_μ

$$-igJ^3_\mu W^{3,\mu} - i\, g'/2\, J^Y_\mu B^\mu$$

$$-i\left[g \sin\vartheta_W J^3_\mu + g' \cos\vartheta_W \frac{1}{2} J^Y_\mu \right] A^\mu + \qquad \left(\equiv -ieJ^{EM}_\mu A^\mu \right)$$

$$-i\left[g \cos\vartheta_W J^3_\mu - g' \sin\vartheta_W \frac{1}{2} J^Y_\mu \right] Z^\mu \qquad \left(\equiv -i \frac{g}{\cos\vartheta_W} J^{NC}_\mu Z^\mu \right)$$

The Electromagnetic Current is a linear combination of the 3rd component of the weak isospin current W^0 and of the weak hypercharge current Y

$$eJ^{EM}_\mu = e\left(J^3_\mu + \frac{1}{2} J^Y_\mu \right) \Rightarrow g \sin\vartheta_W = e = g' \cos\vartheta_W \qquad \tan\vartheta_W = g'/g$$

The Neutral Current (Z^0) $$\frac{g}{\cos\vartheta_W} J^{NC}_\mu = g \cos\vartheta_W J^3_\mu - g \frac{\sin^2\vartheta_W}{\cos\vartheta_W} \frac{1}{2} J^Y_\mu$$

$$= g\left(\cos\vartheta_W J^3_\mu - \frac{\sin^2\vartheta_W}{\cos\vartheta_W} J^{EM}_\mu + \frac{\sin^2\vartheta_W}{\cos\vartheta_W} J^3_\mu \right)$$

$$J^{NC}_\mu = J^3_\mu - \sin^2\vartheta_W J^{EM}_\mu$$

$SU(2)_L \times U(1)_Y$ Lagrangian

In order to construct an electro-weak theory we start from the "free" Dirac Lagrangian

$$L = \bar{\psi}\left(i\gamma_\mu \partial^\mu - m\right)\psi$$

and make it invariant under local $SU(2)_L$ and $U(1)_Y$ transformations.
Note that the group SU(2) is non abelian (like SU(3)), while U(1) is abelian.

The left-handed fields ψ_L enter in all interactions, electromagnetic and weak.
We require the ψ_L fields to be invariant under local $SU(2)_L \times U(1)_Y$ transformations:

$$\psi_L \rightarrow \psi'_L = e^{i\alpha_i(x)T_i} e^{i\beta(x)Y/2} \psi_L$$

The right-handed fields, on the other hand, are required to be invariant under $U(1)_Y$ only:

$$\psi_R \rightarrow \psi'_R = e^{i\beta(x)Y/2} \psi_R$$

The operators T and Y are the generators of the $SU(2)_L$ and $U(1)_Y$ groups.
They act on "properties" of the fields that we call weak isospin and weak hypercharge (in analogy to the spin and the electric charge).

commutation rules $\left[T_i, T_j\right] = i\varepsilon_{ijk} T_k$ and $\left[T_i, Y\right] = 0$

The requirement that a field theory is gauge invariant under a particular symmetry group strictly fixes the form or the interaction and the number of gauge bosons.
To restore local gauge invariance we introduce the covariant derivative involving four new gauge fields W^i and B^0

$$i\partial^\mu \to iD^\mu = i\partial^\mu - g\vec{T}\cdot\vec{W}^\mu - g'\frac{1}{2}YB^\mu \qquad \vec{\tau}\cdot\vec{W}^\mu = \begin{pmatrix} W^3 & W^1 - iW^2 \\ W^1 + iW^2 & -W^3 \end{pmatrix}^\mu$$

with g and g'/2 the coupling constants, and T^a the generators of the $SU(2)_L$ and Y the generator of $U(1)_Y$ gauge groups. This leads to the following interaction term

$$L_{int} = -ig\bar{\psi}_L\gamma_\mu\vec{T}\cdot\vec{W}^\mu\psi_L - ig'\bar{\psi}\gamma_\mu\frac{Y}{2}B^\mu\psi$$

The first term involves only left-handed fields, the second term involves left-handed and right-handed fields.
In terms of the currents interacting with the vector fields, L_{int} can be expressed as

$$L_{int} = -ig\vec{J}_\mu\cdot\vec{W}^\mu - ig'\frac{1}{2}J_\mu^Y B^\mu$$

with J_μ the weak isospin current triplet $\vec{J}_\mu = \bar{\psi}_L\gamma_\mu\frac{1}{2}\vec{\tau}\psi_L$

and J_μ^Y the weak hypercharge current $j_\mu^Y = \bar{\psi}\gamma_\mu Y\psi$

Gauge Terms

To complete the Lagrangian we have to add the kinetic terms describing the gauge fields

$$L_{gauge} = -\frac{1}{4}\vec{W}^{\mu\nu} \cdot \vec{W}_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}$$

where the field strength tensors for $SU(2)_L$ and U(1) are

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\varepsilon_{ijk}W^j_\mu W^k_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

last term from gauge invariance requirement

Recall that the $SU(2)_L$ symmetry group is non Abelian, which leads to the self coupling terms between W gauge bosons (triple and quartic vertices).
Moreover, to restore the gauge invariance, the gauge fields must transform as

$$W^a_\mu(x) \to W^a_\mu(x) - \frac{1}{g}\partial_\mu \alpha^a(x) - i\varepsilon_{ijk}\alpha^j(x)W^k_\mu$$

$$B_\mu(x) \to B_\mu(x) - \frac{1}{g'}\partial_\mu \beta(x)$$

For the moment we will ignore that the W and B gauge bosons are massive (Glashow)

The NC Interaction Terms

The neutral component of the interaction term can be expressed in terms of the A_μ and Z_μ fields as

$$L_{int}^{NC} = -igJ_\mu^3 W^{3,\mu} - ig'\frac{1}{2}J_\mu^Y B^\mu = ieJ_\mu^{em}A^\mu - \frac{ie}{\sin\vartheta_W \cos\vartheta_W}\left[J_\mu^3 - \sin^2\vartheta_W J_\mu^{em}\right]Z^\mu$$

The requirement that the electromagnetic interaction must appear in the Lagrangian fixes the coupling constants

$$e = g\sin\vartheta_W = g'\cos\vartheta_W$$

The interaction term for the charged currents is given by

$$L_{int}^{CC} = -ig\left(J_\mu^- W^{-\,\mu} + J_\mu^+ W^{+\,\mu}\right)$$

with the weak isospin charged currents

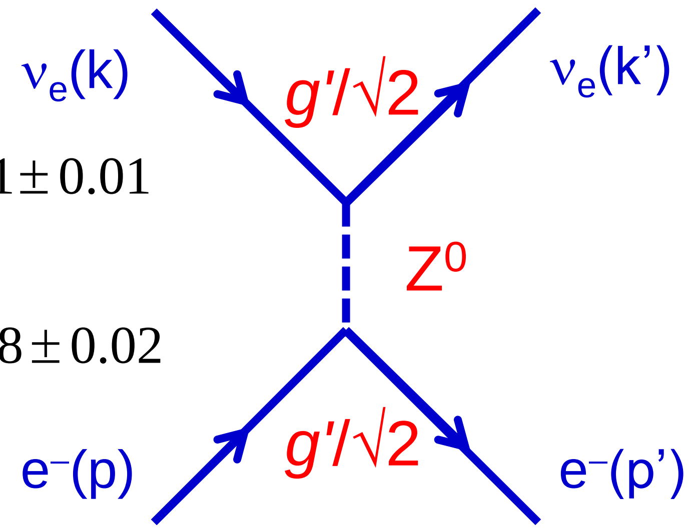
$$J_\mu^\pm = \left(J_\mu^1 \pm iJ_\mu^2\right) = \bar{\chi}_L \gamma_\mu \tau_\pm \chi_L = \left(\bar{\nu}, \bar{e}\right)\gamma_\mu \frac{1}{2}\left(1-\gamma_5\right)\tau_\pm \begin{pmatrix}\nu \\ e\end{pmatrix}$$

and the gauge bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \pm iW_\mu^2\right)$$

Neutral Currents

1973 experimental birth of Standard Model



$\bar{\nu}_\mu e \to \bar{\nu}_\mu e$

$\nu_\mu N \to \nu_\mu X$

$\bar{\nu}_\mu N \to \bar{\nu}_\mu X$

$$R_\nu = \frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} = \frac{\sigma(\nu_\mu N \to \nu_\mu X)}{\sigma(\nu_\mu N \to \mu^- X)} \approx 0.31 \pm 0.01$$

$$R_{\bar{\nu}} = \frac{\sigma^{NC}(\bar{\nu})}{\sigma^{CC}(\bar{\nu})} = \frac{\sigma(\bar{\nu}_\mu N \to \bar{\nu}_\mu X)}{\sigma(\bar{\nu}_\mu N \to \mu^+ X)} \approx 0.38 \pm 0.02$$

First evidence of a weak neutral current

NC anticipated by Glashow in 1961
Until then no weak neutral current effects have been observed.
Note: no flavor change at the vertex, NC conserve flavor!

Very stringent limits on (flavor changing) neutral currents by the absence of decays

$$K^0 \to \mu^+\mu^- \qquad BR = 7\times10^{-9}$$

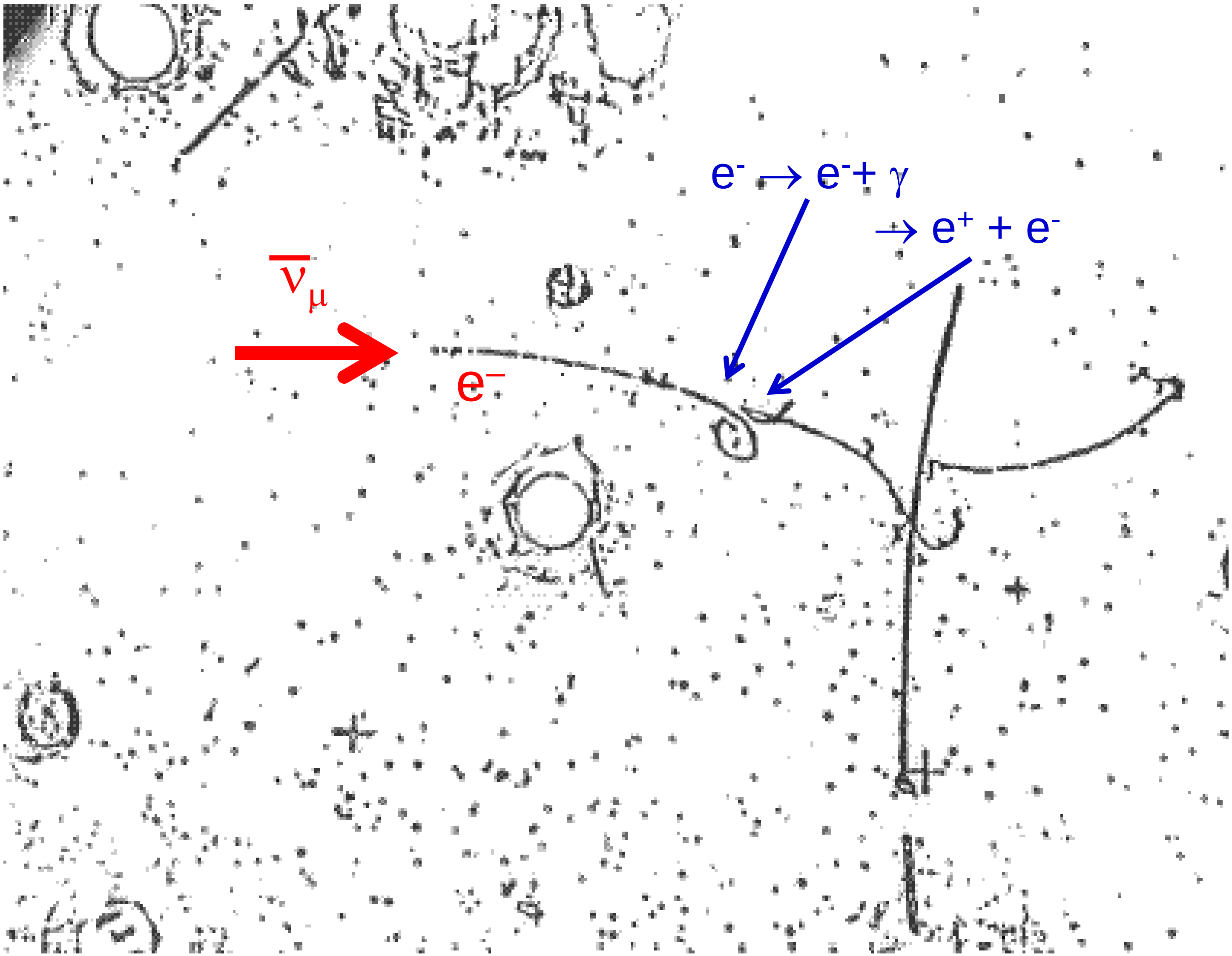
$$K^+ \to \pi^+\mu^+\mu^- \qquad BR < 4\times10^{-11}$$

These small (non-zero!) branching ratios explained well by SM (GIM mechanism), also:

$$B_s^0 \to \mu^+\mu^- \qquad BR = 3\times10^{-9}$$

However in νe, νq scattering NC events are as abundant as CC events,
difficult to detect isolated electron, study on nuclear targets.

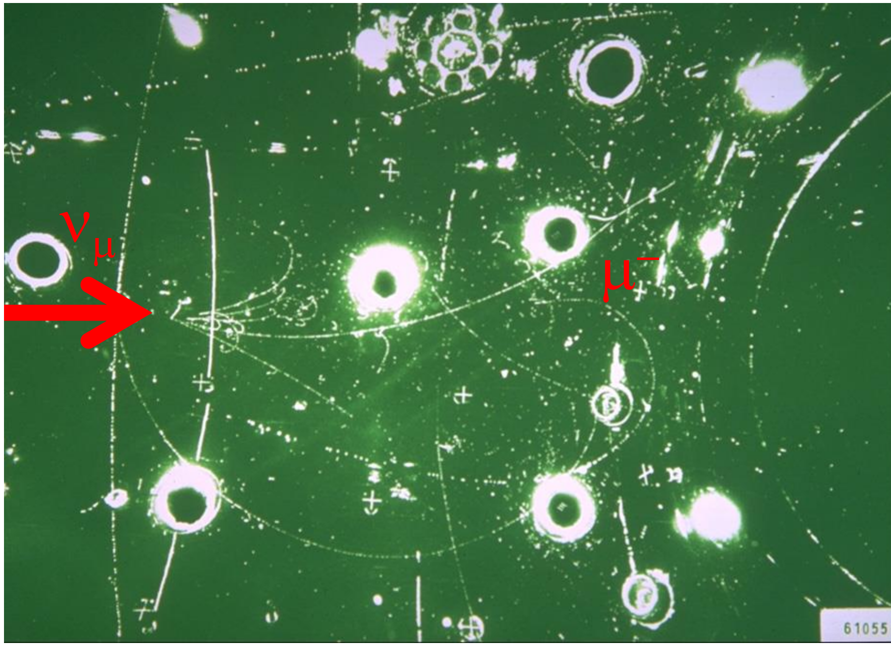
Neutral Currents: $\overline{\nu}_\mu$ e- $\rightarrow \overline{\nu}_\mu$ e- Scattering

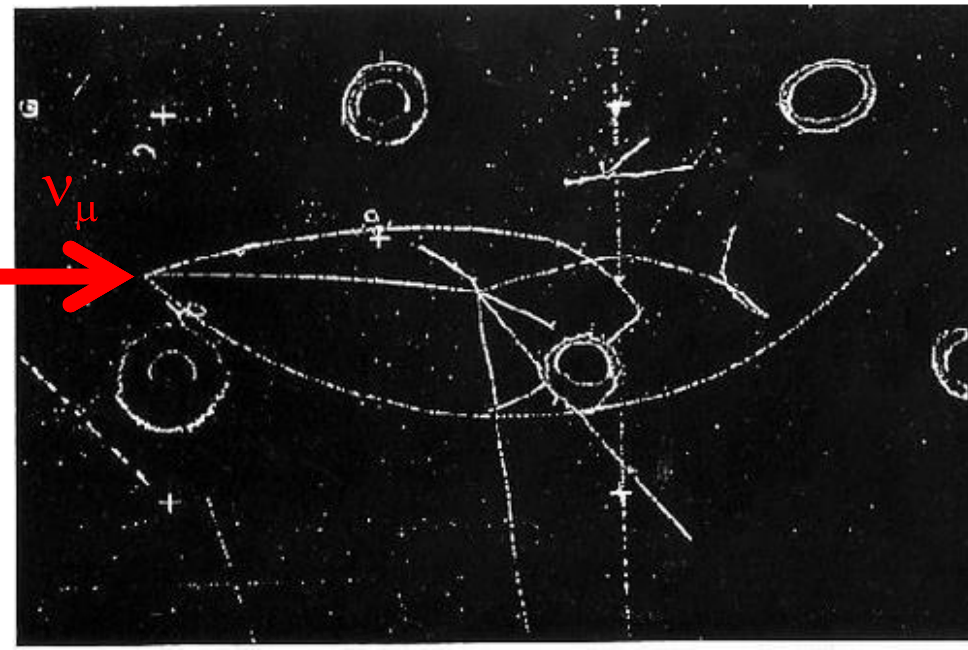


CC and NC – νN Scattering

CC: $\nu_\mu N \to \mu^- + X$

NC: $\nu_\mu N \to \nu_\mu + X$





one lepton (μ^-) detected
all other particles identified as hadrons

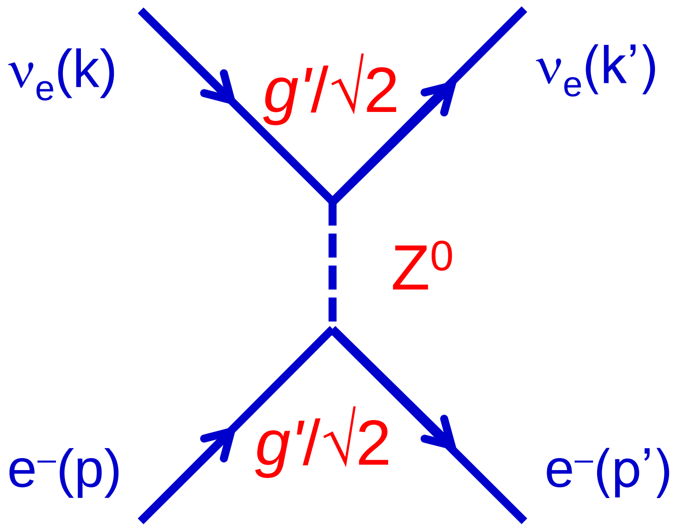
all particles identified as hadrons
no leptons detected!

$$R_\nu = \frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} = \frac{\sigma(\nu_\mu N \to \nu_\mu X)}{\sigma(\nu_\mu N \to \mu^- X)} \approx 0.31 \pm 0.01$$

$$R_{\bar{\nu}} = \frac{\sigma^{NC}(\bar{\nu})}{\sigma^{CC}(\bar{\nu})} = \frac{\sigma(\bar{\nu}_\mu N \to \bar{\nu}_\mu X)}{\sigma(\bar{\nu}_\mu N \to \mu^+ X)} \approx 0.38 \pm 0.02$$

almost as abundant as CC

NC Scattering Amplitude



Develop in analogy to CC at low $q^2 << M_Z^2$

A priori:

i) not necessarily pure V – A, what structure?

ii) can have right handed components (not for ν)
try $c_V V - c_A A$ (c_V and c_A from experiment)

iii) new coupling g', new massive neutral boson

iv) no flavor change at the interaction vertex $\delta_{ff'}$

$$iM^{NC} = -\frac{g'}{\sqrt{2}}\left(\bar{u}_e\gamma^\mu \frac{1}{2}(c_V^e - c_A^e\gamma^5)u_e\right)\frac{g_{\mu\nu} - q_\mu q_\nu / M_Z^2}{q^2 - M_Z^2}\frac{g'}{\sqrt{2}}\left(\bar{u}_\nu\gamma_\mu \frac{1}{2}(c_V^\nu - c_A^\nu\gamma^5)u_\nu\right)$$

effective 4-fermion theory as for CC with new coupling constant $G_{NC} / \sqrt{2} = g'^2 / 8 M_Z^2$
and $c_V^\nu = c_A^\nu = 1/2$ (neutrinos are left-handed) [in a V + A theory $c_V^\nu = -c_A^\nu = 1/2$]

$$M^{NC} = \frac{4G_{NC}}{\sqrt{2}} 2\underbrace{\left(\bar{u}_e\gamma^\mu \frac{1}{2}(c_V^e - c_A^e\gamma^5)u_e\right)}_{(J^{NC})^\mu(e)}\frac{1}{2}\underbrace{\left(\bar{u}_\nu\gamma_\mu \frac{1}{2}(1 - \gamma^5)u_\nu\right)}_{(J^{NC})_\mu(\nu)}$$

neutrino neutral current $$J^{NC}{}_{\mu}(\nu) = \frac{1}{2}\left[\bar{u}_{(\nu)}\ \gamma_{\mu}\ \frac{1}{2}(1-\gamma^5)\ u_{(\nu)}\right]$$

electron neutral current $$J^{NC\mu}(e) = \left[\bar{u}_{(e)}\ \gamma^{\mu}\ \frac{1}{2}(c_V^e - c_A^e\gamma^5)\ u_{(e)}\right]$$

"point-like" interaction of two neutral currents $(J^{NC})^{\mu}$ (e) and $(J^{NC})_{\mu}$ (ν)

$$M^{NC} = \frac{4G_F}{\sqrt{2}} 2\rho J_{\mu}^{NC}(e) J^{NC\mu}(\nu) \qquad \rho = \frac{G_{NC}}{G_F} \approx 1.010 \pm 0.015 = 1 \quad \text{(SM)}$$

ρ determines the relative strength of NC to CC, in the SM $\rho = 1$

In the SM all c_V^i and c_A^i are given in terms of one parameter,

the electroweak mixing Weinberg angle θ_W

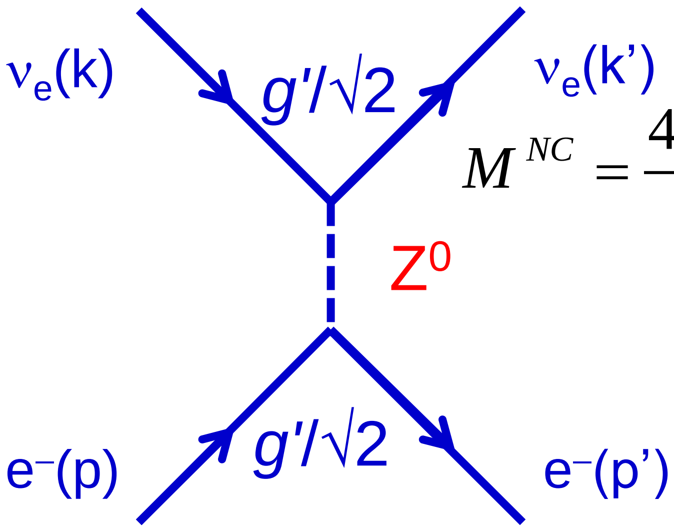
$$\tan\vartheta_W = g'/g \qquad e = g\cdot\sin\vartheta_W = g'\cdot\cos\vartheta_W$$

θ_W measures the relative strength of CC and NC couplings with $\rho = \frac{M_W^2}{M_Z^2\ \cos^2\vartheta_W} = 1$

$c_V^e = -1/2 + 2\sin^2\theta_W \qquad c_A^e = -1/2$

In summary, we have a basis for calculating NC amplitudes.
From now on, assume $\rho = 1$ and $G_{NC} = G_F$. The only unknowns are c_V^e and c_A^e .

NC Couplings c_V and c_A



Recall the NC transition amplitude (L10)

$$M^{NC} = \frac{4G_{NC}}{\sqrt{2}} 2 \underbrace{\left(\bar{e}\, \gamma^{\mu} \frac{1}{2}(c_V^e - c_A^e \gamma^5)\; e \right)}_{(J^{NC})^{\mu}\,(e)} \frac{1}{2} \underbrace{\left(\bar{\nu}\, \gamma_{\mu} \frac{1}{2}(1-\gamma^5)\; \nu \right)}_{(J^{NC})_{\mu}\,(\nu)}$$

where we have introduced the neutral current couplings c_A and c_V to allow for a RH component for the electron field.

Rewrite the Neutral Current interaction as

$$-ig'\left(J_{\mu}^{3} - \sin^2\vartheta_W \; j_{\mu}^{EM} \right) Z^{\mu} =$$

$$-ig'\left(\bar{\psi}\, \gamma_{\mu} \frac{1}{2}(1-\gamma^5) T_3\, \psi \right) Z^{\mu} + ig'\left(\bar{\psi}\, \gamma_{\mu} \sin^2\vartheta_W \; Q\, \psi \right) Z^{\mu} =$$

$$-ig'\bar{\psi}\, \gamma_{\mu} \left[\frac{1}{2}(1-\gamma^5) T_3 - \sin^2\vartheta_W \; Q \right] \psi \; Z^{\mu}$$

and compare to $-ig'\bar{e}\, \gamma_{\mu} \frac{1}{2}(c_V - c_A\gamma^5)\; e$ ⟶

$$\begin{cases} \frac{1}{2} c_V = \frac{1}{2} T_3 - \sin^2\vartheta_W Q \\ \frac{1}{2} c_A = \frac{1}{2} T_3 \end{cases}$$

$$c_R^e = c_V^e - c_A^e \qquad c_L^e = c_V^e + c_A^e$$

c_V and c_A

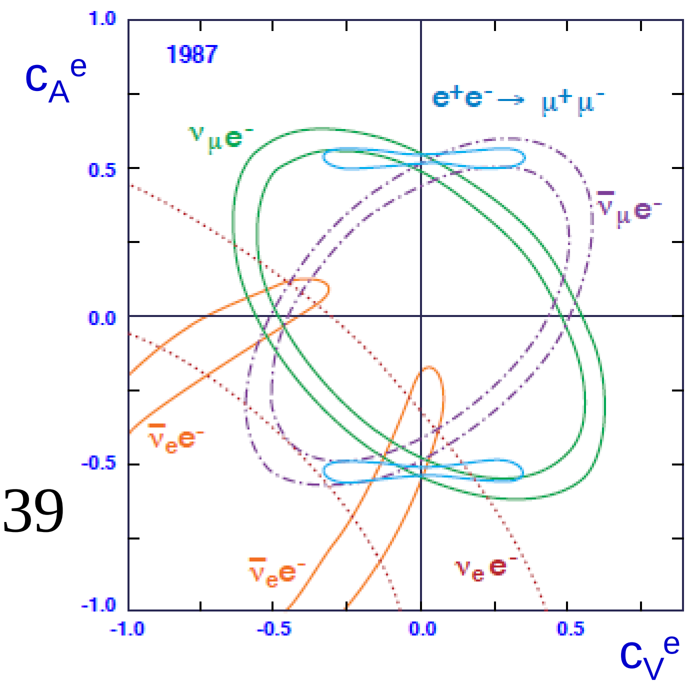
$$c_A^f = T_3^f$$
$$c_V^f = T_3^f - 2\sin^2\vartheta_W\, Q_f$$

	Q	c_A^f	c_V^f
$\nu,\ \bar{\nu}$	0	$+1/2$	$+1/2$
$e^-,\ \mu^-,\ \tau^-$	-1	$-1/2$	$-1/2+2\sin^2\vartheta_W$
$u,\ c,\ t$	$2/3$	$+1/2$	$+1/2-4/3\sin^2\vartheta_W$
$d,\ s,\ b$	$-1/3$	$-1/2$	$-1/2+2/3\sin^2\vartheta_W$

c_V and c_A determined from νe elastic scattering and e^+e^- annihilation at Z pole, which allows also to determine $\sin^2\theta_W$

$$c_V^e = -0.03772 \pm 0.00041$$
$$c_A^e = -0.50117 \pm 0.00027$$
$$c_V^\nu = c_A^\nu = 0.50085 \pm 0.00075$$

$$c_R^e = c_V^e - c_A^e \approx +0.463 \qquad c_L^e = c_V^e + c_A^e \approx -0.539$$
$$c_R^\nu = c_V^\nu - c_A^\nu = 0 \qquad c_L^\nu = c_L^\nu + c_L^\nu = 1$$



Electroweak unification achieved if g = e

In reality e = g sin θ_W
~0.3 = ~0.6 ~0.5

$\sin^2\theta_W(M_Z) = 0.23126 \pm 0.00005$

NC $\nu_e e^- \to \nu_e e^-$ Cross Sections

To start, let's consider $\nu_\mu e^-$ or $\nu_\tau e^-$ scattering (no CC channel!). The NC amplitude is

$$M^{NC}(\nu_\mu e^- \to \nu_\mu e^-) = \frac{4G_F}{\sqrt{2}} 2\rho \left(\bar{u}_e \gamma^\mu \tfrac{1}{2}(c_V^e - c_A^e \gamma^5) u_e \right) \frac{1}{2} \left(\bar{u}_\nu \gamma_\mu \tfrac{1}{2}(1-\gamma^5) u_\nu \right)$$

Using the CC current results

$$\frac{d\sigma(\nu_e e^- \to e^- \nu_e)}{dy} = \frac{G_F^2}{\pi} s \quad \text{and} \quad \frac{d\sigma(\bar{\nu}_e e^- \to \bar{\nu}_e e^-)}{dy} = \frac{G_F^2}{\pi} s(1-y)^2$$

("left-handed") ("right-handed")

we obtain directly

$$\frac{d\sigma^{NC}(\nu_\mu e^- \to \nu_\mu e^-)}{dy} = \frac{G_F^2 s}{4\pi} \Big[\underbrace{(c_V^e + c_A^e)^2}_{c_L} + \underbrace{(c_V^e - c_A^e)^2}_{c_R} (1-y)^2 \Big]$$

and after integrating over y (or d cosθ)

$$\sigma^{NC}(\nu_\mu e^- \to \nu_\mu e^-) = \frac{G_F^2 s}{3\pi} \left[\left(c_V^e\right)^2 + c_V^e c_A^e + \left(c_A^e\right)^2 \right]$$

$$\sigma^{NC}(\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-) = \frac{G_F^2 s}{3\pi} \left[\left(c_V^e\right)^2 - c_V^e c_A^e + \left(c_A^e\right)^2 \right]$$

Finally, we can derive the full $\nu_e e^-$ scattering amplitude!
Both the CC (W exchange) and NC (Z exchange) channels contribute:
add the amplitudes $M = M^{CC}(\nu_e e^- \to e^- \nu_e) + M^{NC}(\nu_e e^- \to \nu_e e^-)$

$$M(\nu_e e^- \to \nu_e e^-) = \frac{4G_F}{\sqrt{2}}\left(\bar{u}_e\, \gamma^\mu\, \tfrac{1}{2}(1-\gamma^5)u_e\right)\left(\bar{u}_\nu\, \gamma_\mu\, \tfrac{1}{2}(1-\gamma^5)u_\nu\right) + \quad \textbf{CC}$$

$$\frac{4G_F}{\sqrt{2}}\, 2\rho\left(\bar{u}_e\, \gamma^\mu\, \tfrac{1}{2}(c_V^e - c_A^e\gamma^5)u_e\right)\frac{1}{2}\left(\bar{u}_\nu\, \gamma_\mu\, \tfrac{1}{2}(1-\gamma^5)u_\nu\right) \quad \textbf{NC}$$

Adding the amplitudes ($\rho = 1$ and $G_{NC} = G_F$)

$$M(\nu_e e^- \to \nu_e e^-) = \frac{4G_F}{\sqrt{2}}\left(\bar{u}_e\, \gamma^\mu\, \tfrac{1}{2}(c_V^e + 1 - (c_A^e + 1)\gamma^5)u_e\right)\left(\bar{u}_\nu\, \gamma_\mu\, \tfrac{1}{2}(1-\gamma^5)u_\nu\right)$$

(i.e. equivalent to replace $c_V^e \to c_V^e + 1$ and $c_A^e \to c_A^e + 1$ in the NC amplitude)

Putting all together leads to

$$\frac{\mathrm{d}\sigma(\nu_e e^- \to \nu_e e^-)}{\mathrm{d}y} = \frac{G_F^2 s}{4\pi}\left[(c_V^e + c_A^e + 2)^2 + (c_V^e - c_A^e)^2(1-y)^2\right]$$

$$\sigma(\nu_e e^- \to \nu_e e^-) = \frac{G_F^2 s}{4\pi}\left[(c_V^e + c_A^e + 2)^2 + \frac{1}{3}(c_V^e - c_A^e)^2\right]$$

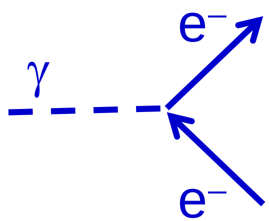
equation of an ellipse in (c_V, c_A)

Feynman Rules (for Leptons)

vertex factor | interaction term

QED

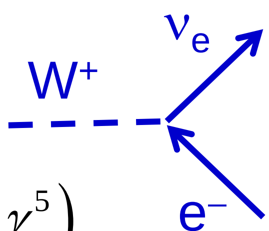
$$-ie\gamma^{\mu}$$



$$-ie\left(\bar{\psi}\,\gamma^{\mu} Q\,\psi\right)A_{\mu}$$

weak
charge raising

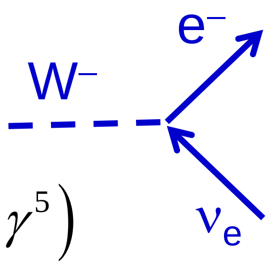
$$-i\frac{g}{\sqrt{2}}\gamma^{\mu}\frac{1}{2}\left(1-\gamma^{5}\right)$$



$$-i\frac{g}{\sqrt{2}}\left(\bar{\chi}_{L}\gamma^{\mu}\frac{1}{2}\tau_{+}\chi_{L}\right)W_{\mu}^{+}$$

weak
charge lowering

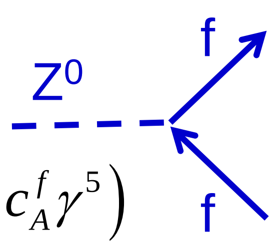
$$-i\frac{g}{\sqrt{2}}\gamma^{\mu}\frac{1}{2}\left(1-\gamma^{5}\right)$$



$$-i\frac{g}{\sqrt{2}}\left(\bar{\chi}_{L}\gamma^{\mu}\frac{1}{2}\tau_{-}\chi_{L}\right)W_{\mu}^{-}$$

weak
neutral

$$-i\frac{g}{\cos\vartheta_{W}}\gamma^{\mu}\frac{1}{2}\left(c_{V}^{f}-c_{A}^{f}\gamma^{5}\right)$$



$$-i\frac{g}{\cos\vartheta_{W}}\bar{\psi}_{f}\gamma^{\mu}\frac{1}{2}\left(c_{V}^{f}-c_{A}^{f}\gamma^{5}\right)\psi_{f}Z_{\mu}=$$

$$-i\frac{g}{\cos\vartheta_{W}}\bar{\psi}_{f}\gamma^{\mu}\left[\frac{1}{2}\left(1-\gamma^{5}\right)T_{3}-\sin^{2}\vartheta_{W}Q\right]\psi_{f}Z_{\mu}$$

Effective Current-Current Interaction

Charged Current interactions can be described with invariant amplitudes of the form (Fermi theory)

$$M^{CC} = \frac{4G_F}{\sqrt{2}} J^{\mu} J_{\mu}^{\dagger}$$

Let the interaction proceed via the exchange of massive charged vector bosons $W^{\pm}$.
First rewrite the basic charged current interaction in the form

$$-i\frac{g}{\sqrt{2}}\left(J^{\mu}W_{\mu}^{+} + J^{\mu\,\dagger}W_{\mu}^{-}\right)$$

then calculate the amplitude using the low q^2 approximation for the W propagator

$$M^{CC} = \left(\frac{g}{\sqrt{2}} J^{\mu}\right) \frac{1}{M_W^2} \left(\frac{g}{\sqrt{2}} J_{\mu}^{\dagger}\right)$$

Comparison of the two gives

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

Similarly for the Neutral Current amplitude in terms of Z exchange express the amplitude

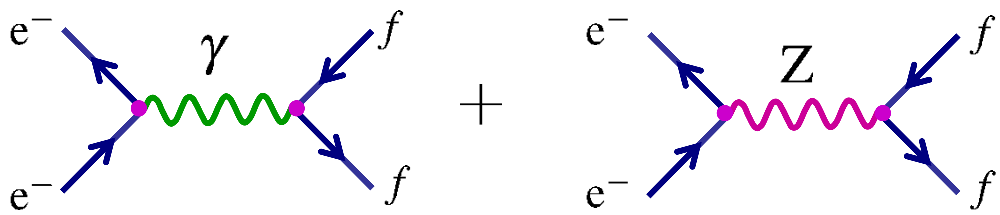
$$M^{NC} = \left(\frac{g}{\cos\vartheta_W} J^{\mu\ NC}\right) \frac{1}{M_Z^2} \left(\frac{g}{\cos\vartheta_W} J_{\mu}^{NC}\right)$$

Comparison with the effective current – current interaction form gives (ρ = NC / CC)

$$\rho\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_Z^2\cos^2\vartheta_W} \quad \Rightarrow \quad \rho = \frac{M_W^2}{M_Z^2\cos^2\vartheta_W} = 1 \quad \text{(SM)}$$

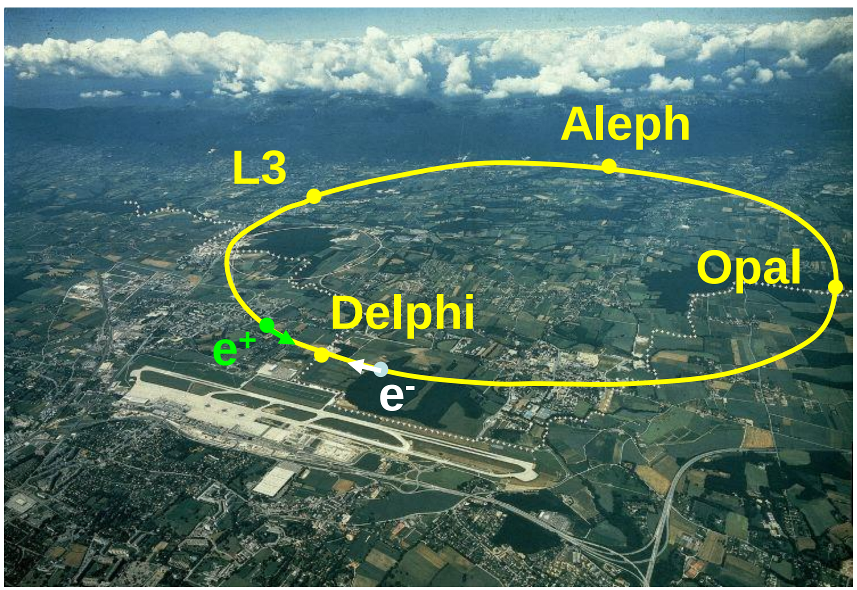
e^+e^- Annihilation

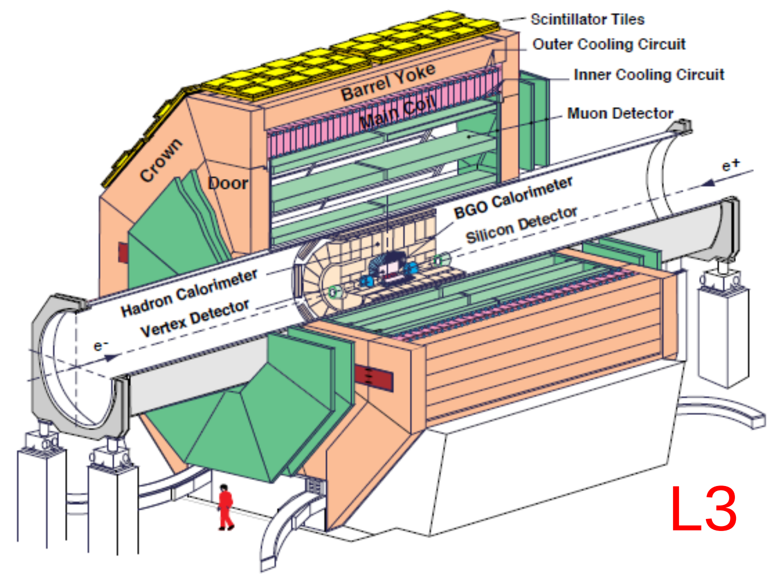
In addition to the electromagnetic interaction (γ), the e^+e^- annihilation can proceed also via the weak neutral interaction (Z).



The final states are indistinguishable $\Rightarrow$ add the amplitudes, which generates an interference between the γ and Z exchange diagrams

$$\sigma_{tot}\left(e^+e^- \to \mu^+\mu^-\right) = \sigma_\gamma + \sigma_Z + \sigma_I$$

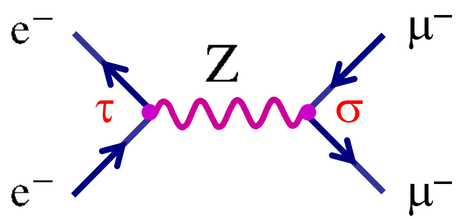




The Z^0 Amplitude

We already studied the electromagnetic interaction

$$iM^{\gamma} = -e^2\left(\bar{\mu}\,\gamma^{\sigma}\mu\right)\frac{g_{\sigma\tau}}{k^2}\left(\bar{e}\,\gamma^{\tau}e\right) \quad\rightarrow\quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\vartheta\right)$$



$e^+ e^-$ vertex $\quad \bar{e}\,\frac{-ig}{\cos\vartheta_W}\gamma^{\tau}\frac{1}{2}\left(c_V^e - c_A^e\gamma^5\right)e$

Z^0 propagator $\quad \frac{-ig_{\sigma\tau} + ik_{\sigma}k_{\tau}/M_Z^2}{k^2 - M_Z^2}$

$\mu^+ \mu^-$ vertex $\quad \bar{\mu}\,\frac{-ig}{\cos\vartheta_W}\gamma^{\sigma}\frac{1}{2}\left(c_V^{\mu} - c_A^{\mu}\gamma^5\right)\mu$

The amplitude for Z exchange is given by

$$iM^Z = -\frac{g^2}{\cos^2\vartheta_W}\left[\bar{\mu}\,\frac{1}{2}\gamma^{\sigma}\left(c_V - c_A\gamma^5\right)\mu\right]\frac{g_{\sigma\tau} - k_{\sigma}k_{\tau}/M_Z^2}{k^2 - M_Z^2}\left[\bar{e}\,\frac{1}{2}\gamma^{\tau}\left(c_V - c_A\gamma^5\right)e\right]$$

Letting $k^2 = s$ and setting

$$c_R = c_V - c_A \quad \text{and} \quad c_L = c_V + c_A \quad\Rightarrow\quad c_V - c_A\gamma^5 = c_R\,\frac{1}{2}\left(1+\gamma^5\right) + c_L\,\frac{1}{2}\left(1-\gamma^5\right)$$

leads to

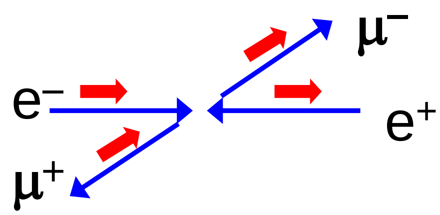
$$iM^Z = -\frac{g^2}{\cos^2\vartheta_W}\left[c_R\left(\bar{\mu}_R\gamma^{\sigma}\mu_R\right) + c_L\left(\bar{\mu}_L\gamma^{\sigma}\mu_L\right)\right]\frac{g_{\sigma\tau}}{s - M_Z^2}\left[c_R\left(\bar{e}_R\gamma^{\tau}e_R\right) + c_L\left(\bar{e}_L\gamma^{\tau}e_L\right)\right]$$

where we have separated the LH components and the RH components.

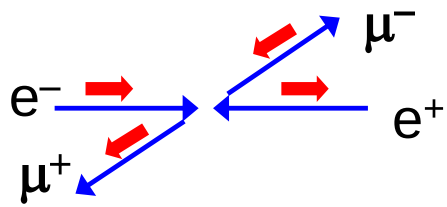
Helicity Decomposition

To calculate $|M_Z|^2$, we have to evaluate 4 terms according to the lepton chiralities (~ helicities, i.e. orientation of spins) [instead of summing over spins, we study each helicity configuration separately)

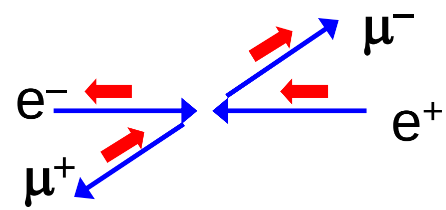
$$M_{RR} = -\frac{g^2}{\cos^2 \vartheta_W}\left[c_R^e \left(\bar{e}_R \gamma^\sigma e_R\right) \frac{g_{\sigma\tau}}{s - M_Z^2} c_R^\mu \left(\bar{\mu}_R \gamma^\tau \mu_R\right)\right]$$



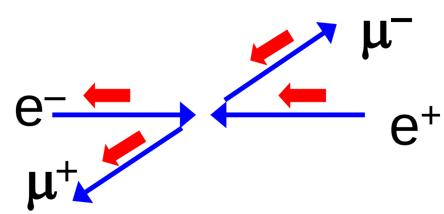
$$M_{RL} = -\frac{g^2}{\cos^2 \vartheta_W}\left[c_R^e \left(\bar{e}_R \gamma^\sigma e_R\right) \frac{g_{\sigma\tau}}{s - M_Z^2} c_L^\mu \left(\bar{\mu}_L \gamma^\tau \mu_L\right)\right]$$



$$M_{LR} = -\frac{g^2}{\cos^2 \vartheta_W}\left[c_L^e \left(\bar{e}_L \gamma^\sigma e_L\right) \frac{g_{\sigma\tau}}{s - M_Z^2} c_R^\mu \left(\bar{\mu}_R \gamma^\tau \mu_R\right)\right]$$



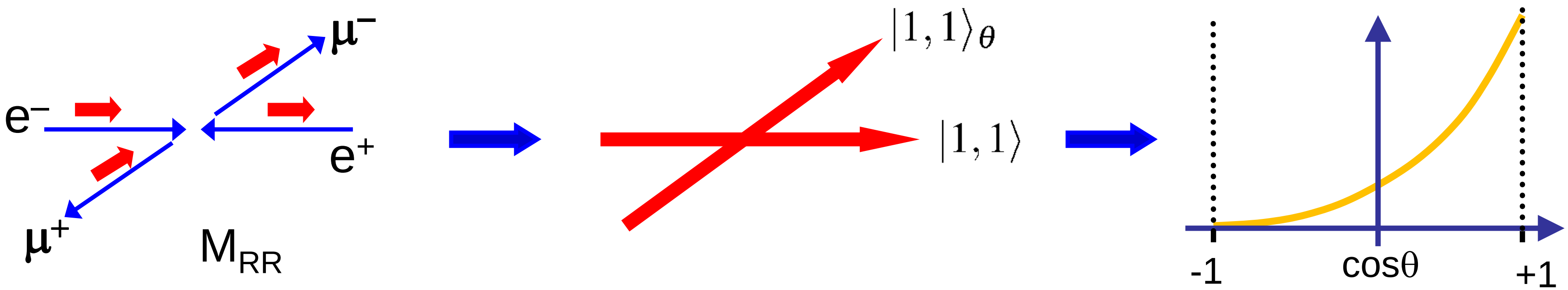
$$M_{LL} = -\frac{g^2}{\cos^2 \vartheta_W}\left[c_L^e \left(\bar{e}_L \gamma^\sigma e_L\right) \frac{g_{\sigma\tau}}{s - M_Z^2} c_L^\mu \left(\bar{\mu}_L \gamma^\tau \mu_L\right)\right]$$



L / R refers to the chirality of the initial / final state fermions.

We have already encounter in QED similar terms,
also $\bar{\nu}\, e \rightarrow \bar{\nu}\, e$ scattering (s-channel, J = 1),
so we can borrow the results from there (angular dependence)

$$\frac{e^2}{q^2} \rightarrow \frac{g^2}{\cos^2\vartheta_W}\,\frac{1}{s - M_Z^2}$$

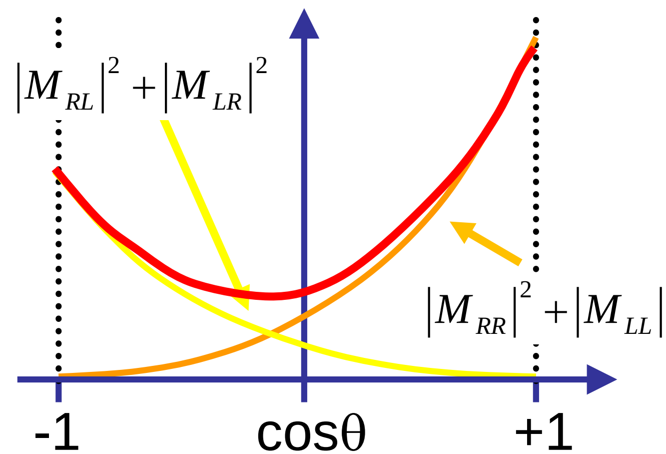


$$|M_{RR}|^2 = \left(\frac{g^2/\cos^2\vartheta_W}{s - M_Z^2}\right)^2 \left(c_R^e\right)^2 \left(c_R^\mu\right)^2 s^2 \left(1 + \cos\vartheta\right)^2$$

$$|M_{LL}|^2 = \left(\frac{g^2/\cos^2\vartheta_W}{s - M_Z^2}\right)^2 \left(c_L^e\right)^2 \left(c_L^\mu\right)^2 s^2 \left(1 + \cos\vartheta\right)^2$$

$$|M_{RL}|^2 = \left(\frac{g^2/\cos^2\vartheta_W}{s - M_Z^2}\right)^2 \left(c_R^e\right)^2 \left(c_L^\mu\right)^2 s^2 \left(1 - \cos\vartheta\right)^2$$

$$|M_{LR}|^2 = \left(\frac{g^2/\cos^2\vartheta_W}{s - M_Z^2}\right)^2 \left(c_L^e\right)^2 \left(c_R^\mu\right)^2 s^2 \left(1 - \cos\vartheta\right)^2$$



$$\cos(\pi - \vartheta) = -\cos\vartheta$$

Around the Z^0 Peak

Need to consider carefully the Z propagator, which diverges for $\sqrt{s} \to M_Z$.

Account also for the fact that the Z boson is an unstable particle (i.e. a resonance) which is equivalent to the replacement in the wave function $M_Z \;\to\; M_Z + i\Gamma_Z/2$ (relativistic Breit – Wigner).

Make same replacement in the propagator, valid if $\Gamma_Z << M_Z$:

$$\left|\frac{1}{s-M_Z^2}\right|^2 \;\to\; \left|\frac{1}{s-M_Z^2+iM_Z\Gamma_Z}\right|^2 = \frac{1}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}$$

For instance, the matrix element $|M_{RR}|^2$ becomes

$$\left|M_{RR}\right|^2 = \frac{\left(g^2/\cos^2\vartheta_W\right)^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}\left(c_R^e\right)^2\left(c_R^\mu\right)^2 s^2\left(1+\cos\vartheta\right)^2$$

giving

$$\frac{\mathrm{d}\sigma_{RR}}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{\left(g^2/\cos^2\vartheta_W\right)^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}\left(c_R^e\right)^2\left(c_R^\mu\right)^2 s^2\left(1+\cos\vartheta\right)^2$$

$$\frac{\mathrm{d}\sigma_{RL}}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{\left(g^2/\cos^2\vartheta_W\right)^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}\left(c_R^e\right)^2\left(c_L^\mu\right)^2 s^2\left(1-\cos\vartheta\right)^2$$

Unpolarized σ_Z

To calculate the σ_Z cross section need to sum over all 4 matrix elements (spin states) and average over the initial spin states.
Assuming unpolarized beams there are 4 combinations of initial electron/positron spins

$$\left\langle \left|M_{fi}\right|^2 \right\rangle = \frac{1}{2}\cdot\frac{1}{2}\left(\left|M_{RR}\right|^2 + \left|M_{LL}\right|^2 + \left|M_{RL}\right|^2 + \left|M_{LL}\right|^2\right)$$

$$= \frac{1}{4}\frac{\left(g^2/\cos^2\vartheta_W\right)^2}{\left(s-M_Z^2\right)^2 + M_Z^2\Gamma_Z^2} s^2 \left\{\left(\left(c_R^e c_R^\mu\right)^2 + \left(c_L^e c_L^\mu\right)^2\right)(1+\cos\vartheta)^2 + \left(\left(c_R^e c_L^\mu\right)^2 + \left(c_L^e c_R^\mu\right)^2\right)(1-\cos\vartheta)^2\right\}$$

The part in {…} can be rearranged as

$$\{\ldots\} = \frac{1}{4}\left[\left(c_V^e\right)^2 + \left(c_A^e\right)^2\right]\left[\left(c_V^\mu\right)^2 + \left(c_A^\mu\right)^2\right]\left(1+\cos^2\vartheta\right) + 2c_V^e c_A^e c_V^\mu c_A^\mu \cos\vartheta$$

and the cross section follows

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left\langle \left|M_{fi}\right|^2 \right\rangle$$

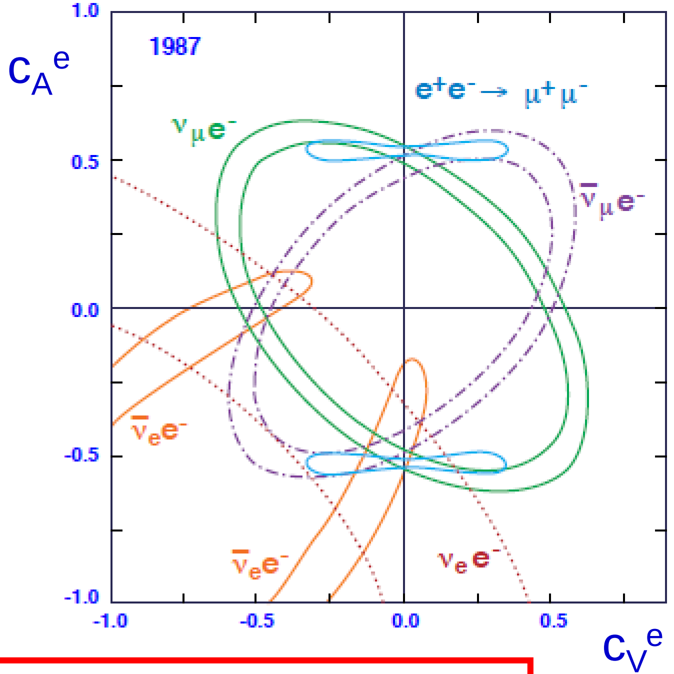
$$= \frac{1}{64\pi^2 s}\frac{1}{4}\frac{\left(g^2/\cos^2\vartheta_W\right)^2}{\left(s-M_Z^2\right)^2 + M_Z^2\Gamma_Z^2} s^2 \left\{\frac{1}{4}\left[\left(c_V^e\right)^2 + \left(c_A^e\right)^2\right]\left[\left(c_V^\mu\right)^2 + \left(c_A^\mu\right)^2\right]\left(1+\cos^2\vartheta\right) + 2c_V^e c_A^e c_V^\mu c_A^\mu \cos\vartheta\right\}$$

Integrating over dΩ gives

$$\sigma_{e^+e^-\to Z\to\mu^+\mu^-} = \frac{1}{192\pi s}\frac{\left(g^2/\cos^2\vartheta_W\right)^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}s^2\left[\left(c_V^e\right)^2+\left(c_A^e\right)^2\right]\left[\left(c_V^\mu\right)^2+\left(c_A^\mu\right)^2\right]$$

The σ_Z cross section is proportional to the sums of the squares of the vector- and axial-vector couplings of the initial and final state fermions

$$\left(c_V^f\right)^2+\left(c_A^f\right)^2$$



This is the additional "relation" which in conjunction with νe scattering experiments allows us to determine c_V and c_A.

For $e^+ e^- \to q\bar{q}$ introduce the corresponding vector- and axial-vector couplings and the color factor $N_C = 3$

$$\sigma_{e^+e^-\to Z\to q\bar{q}} = 3\frac{1}{192\pi s}\frac{\left(g^2/\cos^2\vartheta_W\right)^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}s^2\left[\left(c_V^e\right)^2+\left(c_A^e\right)^2\right]\left[\left(c_V^q\right)^2+\left(c_A^q\right)^2\right]$$

The Interference Term

To derive the interference term between γ and Z amplitudes $\sigma_I \propto 2\,\mathrm{Re}\left[M_\gamma M_Z^\dagger\right]$

let's rewrite the $e^+ e^- \rightarrow \mu^+ \mu^-$ scattering amplitude as

$$M_{h_e h_f}^{e^+e^-\rightarrow f\bar{f}} = s\left(1+h_e h_f \cos\vartheta\right)\left[\frac{4\pi\alpha}{s} + \frac{g^2/\cos^2\vartheta_W \; c_h^e c_h^f}{s - M_Z^2 - iM_Z\Gamma_Z}\right]$$

where h_e is the helicity (~chirality) of the incoming electrons and h_f the helicity of outgoing fermions, q the charge of the outgoing fermions and the c_h's the helicity dependent neutral current weak couplings (c_R or c_L).

The squared amplitude becomes

$$\left|M_{h_e h_f}^{e^+e^-\rightarrow f\bar{f}}\right|^2 = s^2\left(1+h_e h_f \cos\vartheta\right)^2 \left[\left(\frac{4\pi\alpha}{s}\right)^2\right. \qquad \text{γ exchange}$$

$$+\frac{4\pi\alpha}{s}\cdot\frac{g^2/\cos^2\vartheta_W \cdot c_h^e c_h^f \cdot 2\left(s-M_Z^2\right)}{\left(s-M_Z^2\right)^2 + M_Z^2\Gamma_Z^2} \qquad \text{interference}$$

$$\left.+\frac{\left(g^2/\cos^2\vartheta_W\right)^2\left(c_h^e c_h^f\right)^2}{\left(s-M_Z^2\right)^2 + M_Z^2\Gamma_Z^2}\right] \qquad \text{Z exchange}$$

The e^+e^- Cross Section

Putting together all 3 terms, the e^+e^- differential cross section, can be expressed as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left[A_0(s)\left(1+\cos^2\vartheta\right) + A_1(s)\cos\vartheta\right]$$

QED

$$A_0 = 1$$
$$A_1 = 0$$

with

$$A_0(s) = 1 + \frac{1}{2}\mathrm{Re}(r)\left(c_R + c_L\right)^2 + \frac{1}{4}|r|^2\left(c_R^2 + c_L^2\right)^2$$
$$= 1 + 2\,\mathrm{Re}(r)c_V^2 + |r|^2\left(c_V^2 + c_A^2\right)^2$$

$$A_1(s) = \mathrm{Re}(r)\left(c_R + c_L\right)^2 + \frac{1}{2}|r|^2\left(c_R^2 - c_L^2\right)^2$$
$$= \qquad + 4\,\mathrm{Re}(r)c_A^2 + 8|r|^2 c_V^2 c_A^2$$

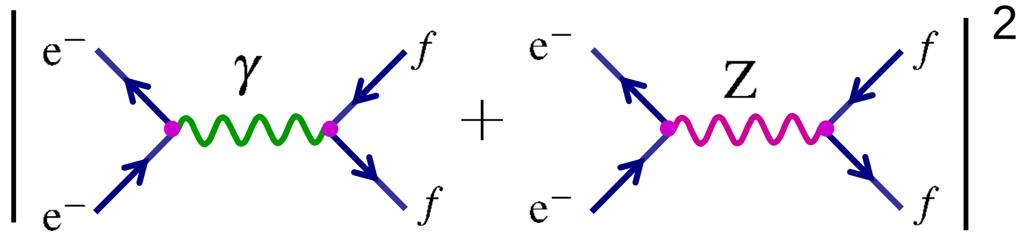
where r “describes” the Breit-Wigner line shape characteristic of the Z^0 resonance (the energy dependence of A_0 and A_1 is contained in r)

$$r = \frac{g^2/\cos^2\vartheta_W}{16\pi\,\alpha\,M_Z^2}\,\frac{sM_Z^2}{s - M_Z^2 + iM_Z\Gamma_Z}$$

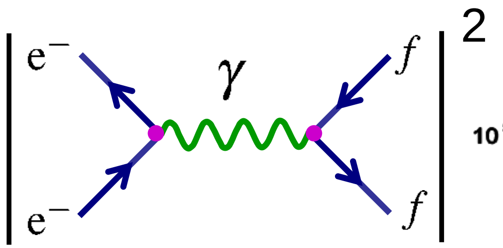
This decomposition of A_0 and A_1 shows the repartition between the electromagnetic term, the interference terms $\propto$ Re(r) and $\propto g^2$, and the week terms $\propto |r|^2$ and $\propto g^4$.

Summary e^+e^- Cross Section

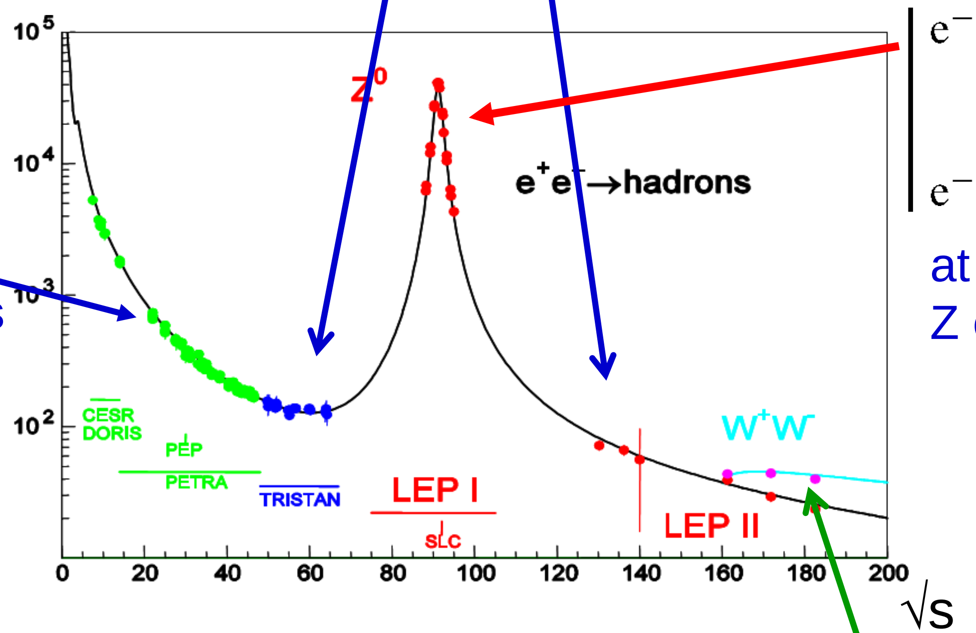
e^+e^- annihilation involves
γ and Z exchange + interference

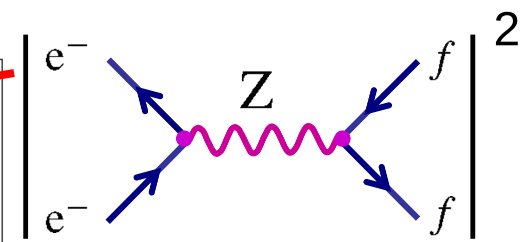


interference



well below Z peak
γ exchange dominates

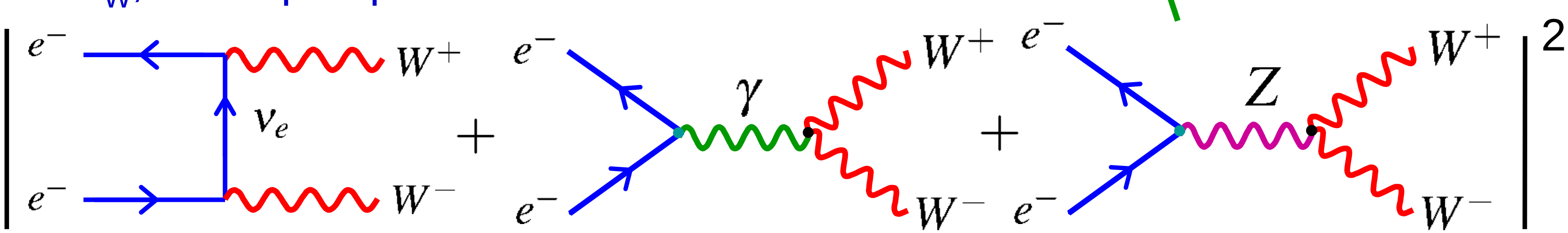




at Z peak
Z exchange dominates

$$\frac{\sigma_Z}{\sigma_\gamma} \approx 200$$

above $2M_W$, W^+W^- pair production dominates



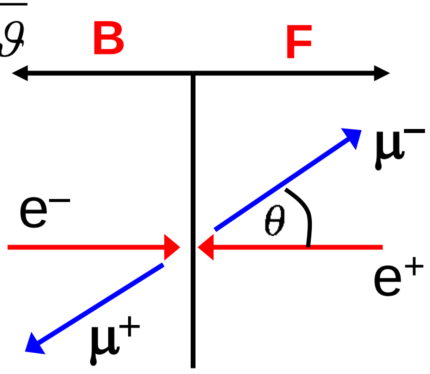
Forward – Backward Asymmetry

QED ($A_0 = 1$ and $A_1 = 0$, valid for $r \to 0$, i.e. well below the Z pole) gives a symmetric angular distribution.

Because $|M_{LL}|^2 + |M_{RR}|^2 \neq |M_{LR}|^2 + |M_{RL}|^2$, the differential cross section is asymmetric, i.e. parity violation (although not maximal as was the case for the W boson).

The weak interaction introduces a Forward – Backward asymmetry

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} \qquad \sigma_F = \int_0^1 \mathrm{d}\cos\vartheta \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\vartheta} \qquad \sigma_B = \int_{-1}^0 \mathrm{d}\cos\vartheta \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\vartheta}$$

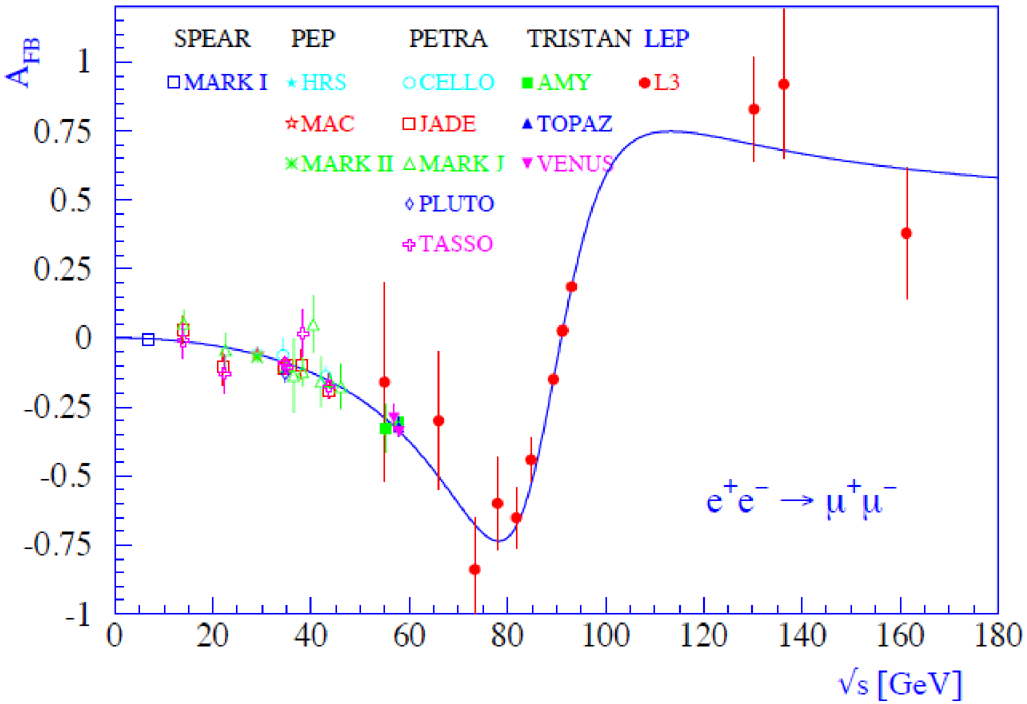


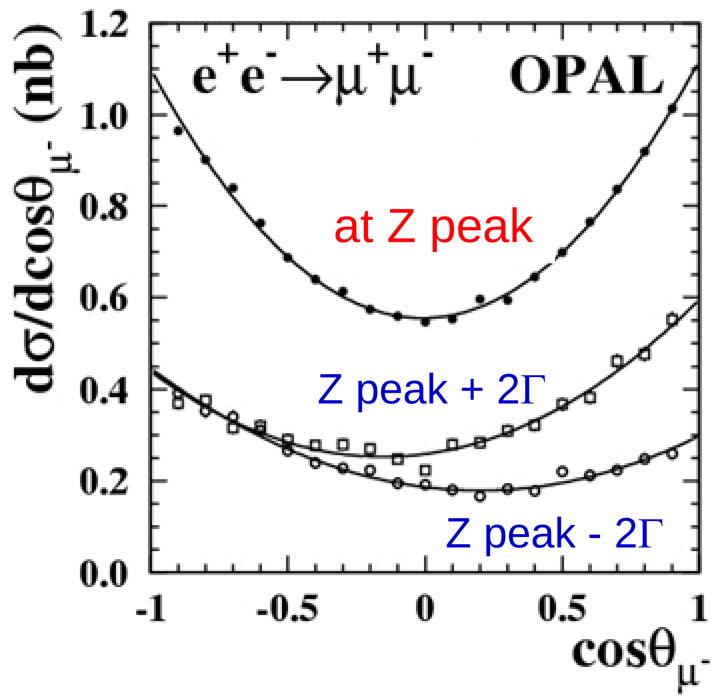
$$\sigma_F = \frac{\alpha^2}{4s}\int_0^1 \mathrm{d}\cos\vartheta\left[A_0\left(1+\cos^2\vartheta\right) + A_1\cos\vartheta\right] = \frac{\alpha^2}{4s}\left(\frac{4}{3}A_0 + \frac{1}{2}A_1\right)$$

$$\sigma_B = \frac{\alpha^2}{4s}\int_{-1}^0 \mathrm{d}\cos\vartheta\left[A_0\left(1+\cos^2\vartheta\right) + A_1\cos\vartheta\right] = \frac{\alpha^2}{4s}\left(\frac{4}{3}A_0 - \frac{1}{2}A_1\right)$$

$$A_{FB} = \frac{A_1}{8/3\,A_0} = \frac{3}{4}\left[\frac{\left(c_L^e\right)^2 - \left(c_R^e\right)^2}{\left(c_L^e\right)^2 + \left(c_R^e\right)^2}\right]\cdot\left[\frac{\left(c_L^\mu\right)^2 - \left(c_R^\mu\right)^2}{\left(c_L^\mu\right)^2 + \left(c_R^\mu\right)^2}\right] = \frac{3}{4}A^e A^\mu = 3\left[\frac{c_V^f c_A^f}{\left(c_V^f\right)^2 + \left(c_A^f\right)^2}\right]^2$$

Observe a non-zero asymmetry because the couplings of the Z to LH and RH fermions are different. Contrast with QED, where the couplings to LH and RH fermions are the same (parity is conserved) and the interaction is F – B symmetric.





$$A^e = 0.1514 \pm 0.0019$$

$$A^\mu = 0.1456 \pm 0.0091$$

$$A^\tau = 0.1449 \pm 0.0040$$

$$A^f = \frac{2c_V^f c_A^f}{\left(c_V^f\right)^2 + \left(c_A^f\right)^2} = 2\frac{c_V^f / c_A^f}{1 + \left(c_V^f / c_A^f\right)^2}$$

The measured F – B asymmetries give the ratio of vector to axial-vector Z couplings. In SM these are related to the weak mixing angle

$$\frac{c_V}{c_A} = \frac{T_3 - 2Q\sin^2\vartheta_W}{T_3} = 1 - 4|Q|\sin^2\vartheta_W \quad \Rightarrow \quad \sin^2\vartheta_W(M_Z^2) = 0.23154 \pm 0.00016$$

Z^0 Width

Around the Z^0 peak can ignore the EM and interference terms, because σ_Z dominates.

Rewrite σ_Z in terms of the Z boson partial decay rates
(use Fermi's golden rule to derive Γ and set $s = M_Z^2$)

$$\Gamma\left(Z \to e^+e^-\right) = \frac{g^2/\cos^2\vartheta_w}{48\pi}\left[\left(c_V^e\right)^2 + \left(c_A^e\right)^2\right] M_Z \quad \text{and} \quad \Gamma\left(Z \to \mu^+\mu^-\right) = \frac{g^2/\cos^2\vartheta_w}{48\pi}\left[\left(c_V^\mu\right)^2 + \left(c_A^\mu\right)^2\right] M_Z$$

as

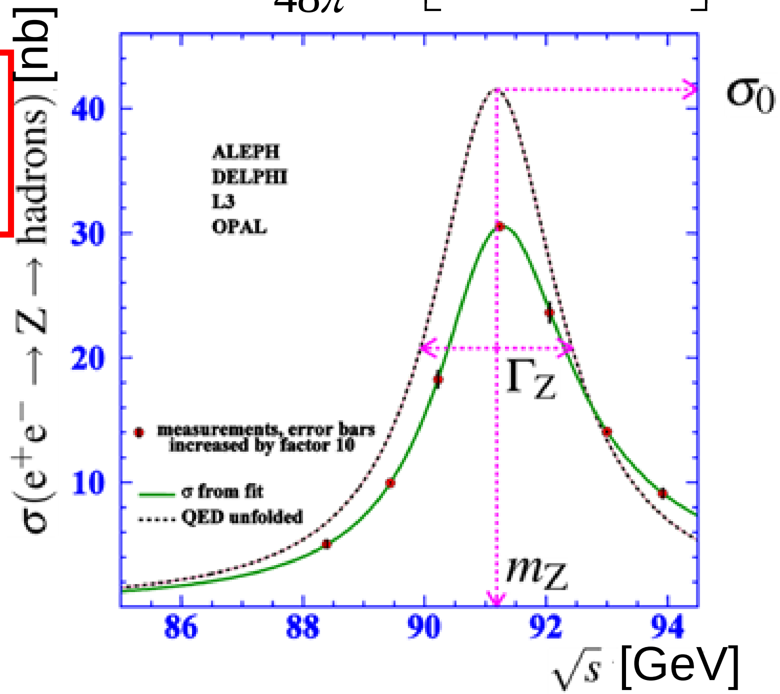
$$\sigma(e^+e^- \to Z \to \mu^+\mu^-) = \frac{12\pi}{M_Z^2}\frac{s}{\left(s - M_Z^2\right)^2 + M_Z^2\Gamma_Z^2}\Gamma_{ee}\Gamma_{\mu\mu}$$

and

$$\sigma\left(e^+e^- \to Z \to f\bar{f}\right) = \frac{12\pi}{M_Z^2}\frac{s}{\left(s - M_Z^2\right)^2 + M_Z^2\Gamma_Z^2}\Gamma_{ee}\Gamma_{ff}$$

with peak cross section equal to

$$\sigma_{f\bar{f}}\left(M_Z^2\right) = \frac{12\pi}{M_Z^2}\frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}$$



which allows the determination of M_Z and Γ_Z (including the partial widths)

$$M_Z = 91.1875 \pm 0.0021 \text{ GeV} \qquad \Gamma_Z = 2.4952 \pm 0.0023 \text{ GeV}$$

Number of Generations

The total decay width is the sum of all partial widths

$$\Gamma_Z = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{had} + N_\nu \Gamma_{\nu\nu} + \ ?$$

Although we cant observe Z decays into neutrinos (invisible decay mode), these decays affect the Z resonance shape for all final states.

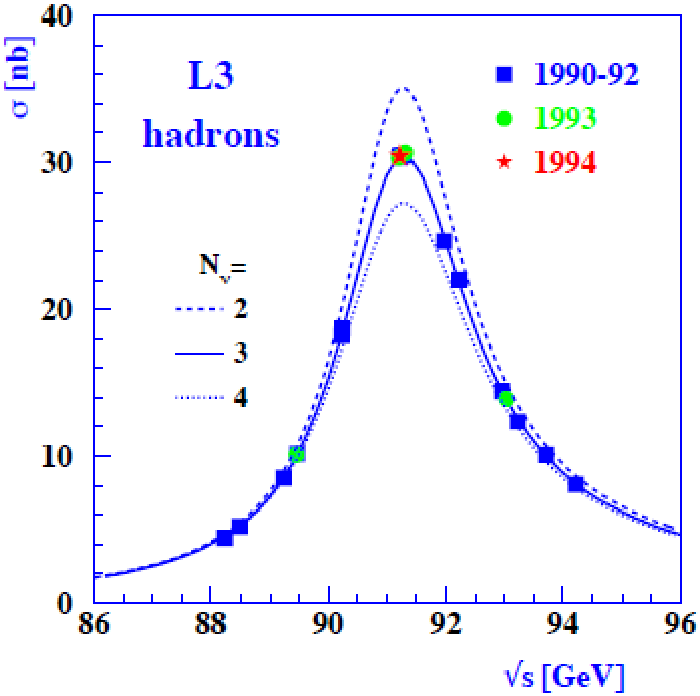
If there were an additional 4th generation would expect $Z \to \nu_4 \bar{\nu}_4$ decays even if the charged fermions were too heavy (as long as $m_\nu < M_Z/2$)

Assuming lepton universality

$$\Gamma_Z = 3\Gamma_{ll} + \Gamma_{had} + N_\nu \Gamma_{\nu\nu}$$

from Z line shape | from peak cross sections | calculated

$$N_\nu = 2.9840 \pm 0.0082$$



The W Bosons

A real (i.e. not virtual) massless spin-1 boson (i.e. the photon) can exist in two transverse polarization states (circular polarization),
although off-mass shell virtual photons can be longitudinally polarized.

A massive spin-1 boson (i.e. the W and Z bosons) acquires also a longitudinal polarization (3rd polarization component).

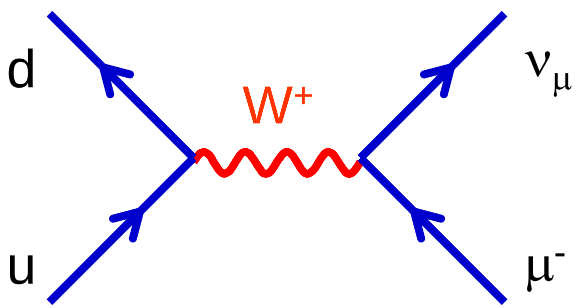
Spin-1 boson wave-functions can be written in terms of the polarization four-vector ε^{μ}:

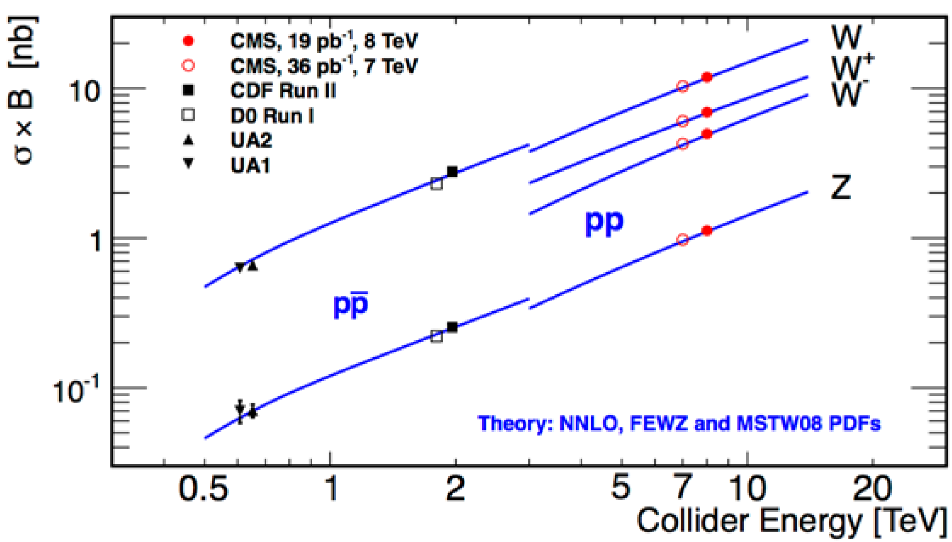
$$W^{\mu} = \varepsilon^{\mu} e^{-ip\cdot x} = \varepsilon^{\mu} e^{i(\vec{p}\cdot\vec{x}-Et)} \qquad \varepsilon^{\mu} p_{\mu} = 0$$

with

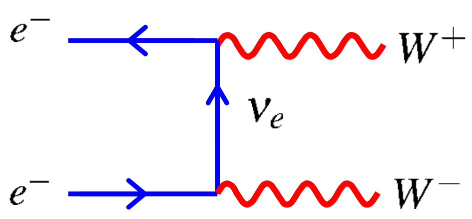
$$\varepsilon^{\mu}_{-} = \frac{1}{\sqrt{2}}(0,1,-i,0) \qquad \varepsilon^{\mu}_{L} = \frac{1}{m}(p_z,0,0,E) \qquad \varepsilon^{\mu}_{+} = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

W bosons can also be produced in $p\bar{p}$ and pp collisions



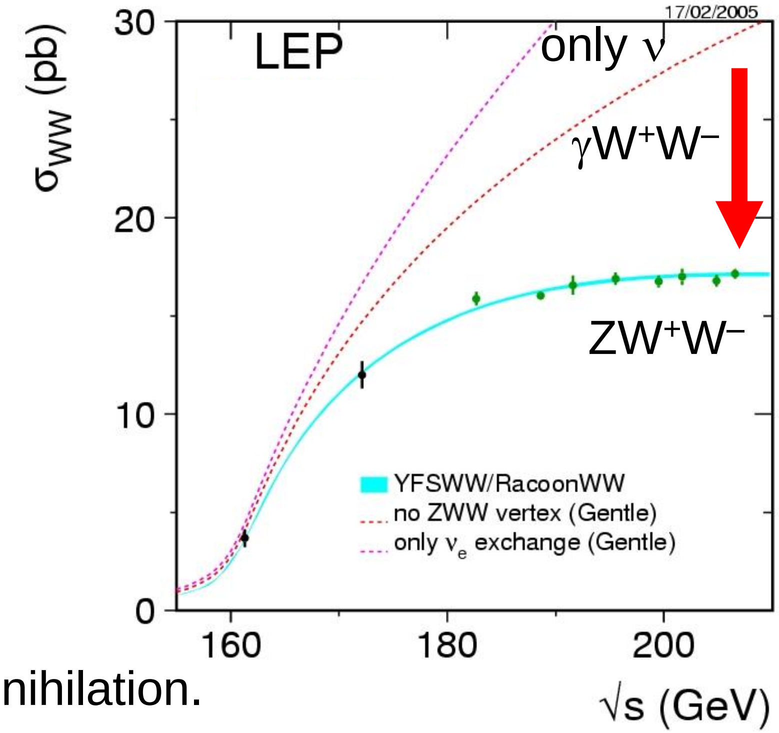


$e^+e^- \rightarrow W^+W^-$ Pair Production



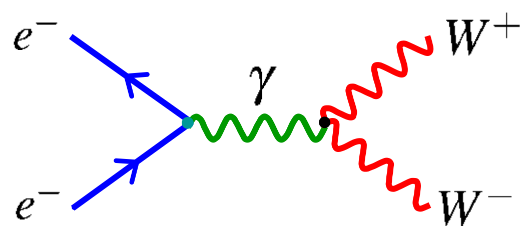
$$\sigma = \frac{\pi\alpha^2 s}{96\sin^4\vartheta_W M_W^2}$$

divergent !

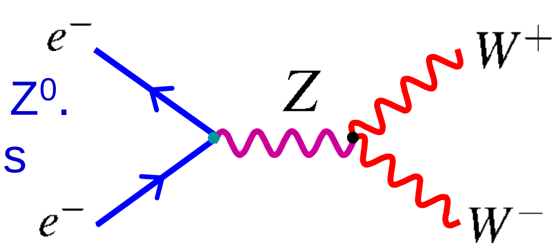


We have a problem however:
the cross section violates unitarity
(i.e. for $s \rightarrow \infty$ the outgoing $W^\pm$ flux is larger than the incoming e^+e^- flux)

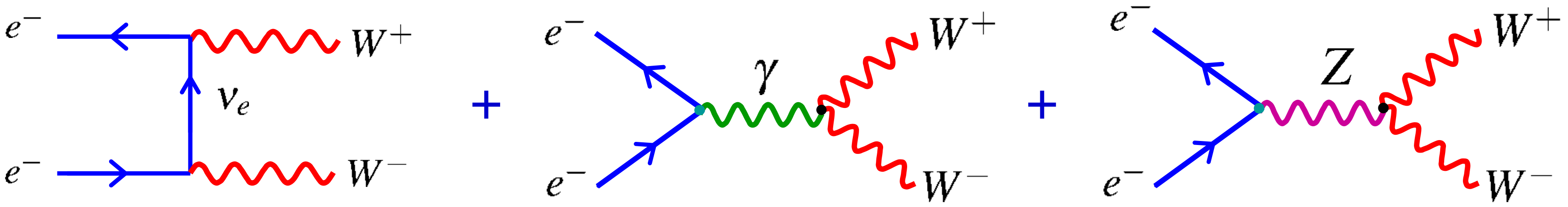
The $W^\pm$ bosons carry electric charge and
W bosons can be produced for example in e^+e^- annihilation.



but the cross section is still divergent.
This problem can be "cured" by introducing a new boson, the Z^0.
The new Z^0 diagram interferes negatively with the W diagrams (almost) solving the unitarity problem.



Add the amplitudes



$$\left|M_{\nu WW} + M_{\gamma WW} + M_{ZWW}\right|^2 < \left|M_{\nu WW} + M_{\gamma WW}\right|^2$$

Only works if Z, γ, W couplings are related $\rightarrow$ Electroweak Unification

Finally, the cross section becomes

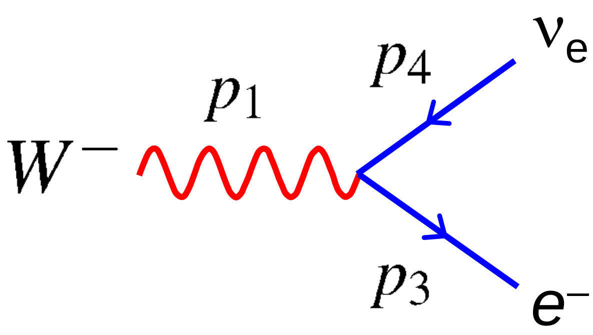
$$\sigma = \frac{\pi\alpha^2}{2\sin^4\vartheta_W}\log\left(s/M_W^2\right)$$

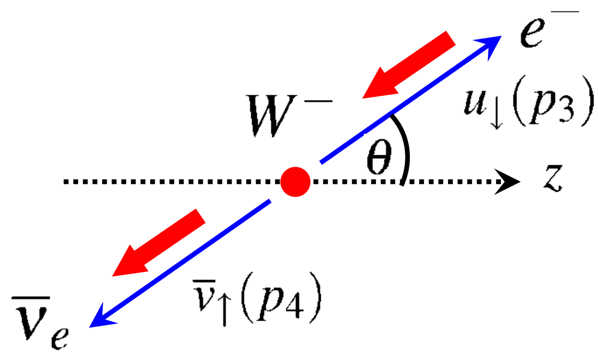
but still exhibits a mild logarithmic divergency.

The presence of the Z (almost) fixes this problem.
Question: What is missing?   Answer: the Higgs boson.

W Decay

in the W center of mass (W rest frame)





The transition rate can be written in terms of the scalar product of the W-boson polarization ε^μ (p_1) and the weak charged current J_μ^- :

$$M_{fi} = \frac{g}{\sqrt{2}} \varepsilon^\mu(p_1) \; \bar{u}(p_3)\gamma_\mu \frac{1}{2}(1-\gamma^5)v(p_4)$$

Take J_μ^- from the helicity decomposition of the $e^+e^- \rightarrow \mu^+\mu^-$ annihilation:

$$J_\mu = 2E(0, +\cos\theta, +i, -\sin\theta)$$

and the polarization four-vectors for a W at rest

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0) \qquad \varepsilon_L^\mu = (0,0,0,1) \qquad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

and calculate separately the matrix elements for the three different polarization states of the W boson with $E = M_W/2$ (ignore the masses of the decay fermions).

Helicity Decomposition

We obtain the following 3 amplitudes for each polarization state of the W boson

$$\varepsilon_- : \quad M_- = \frac{g}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,1,-i,0)M_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}gM_W(1+\cos\theta)$$

$$\varepsilon_L : \quad M_L = \frac{g}{\sqrt{2}}(0,0,0,1)M_W(0,-\cos\theta,-i,\sin\theta) = -\frac{1}{\sqrt{2}}gM_W\sin\theta$$

$$\varepsilon_+ : \quad M_+ = \frac{g}{\sqrt{2}}\frac{-1}{\sqrt{2}}(0,1,i,0)M_W(0,-\cos\theta,-i,\sin\theta) = \frac{1}{2}gM_W(1-\cos\theta)$$

and after taking the modulo squared

$$\varepsilon_- : \quad |M_-|^2 = \frac{1}{4}g^2M_W^2(1+\cos\theta)^2$$

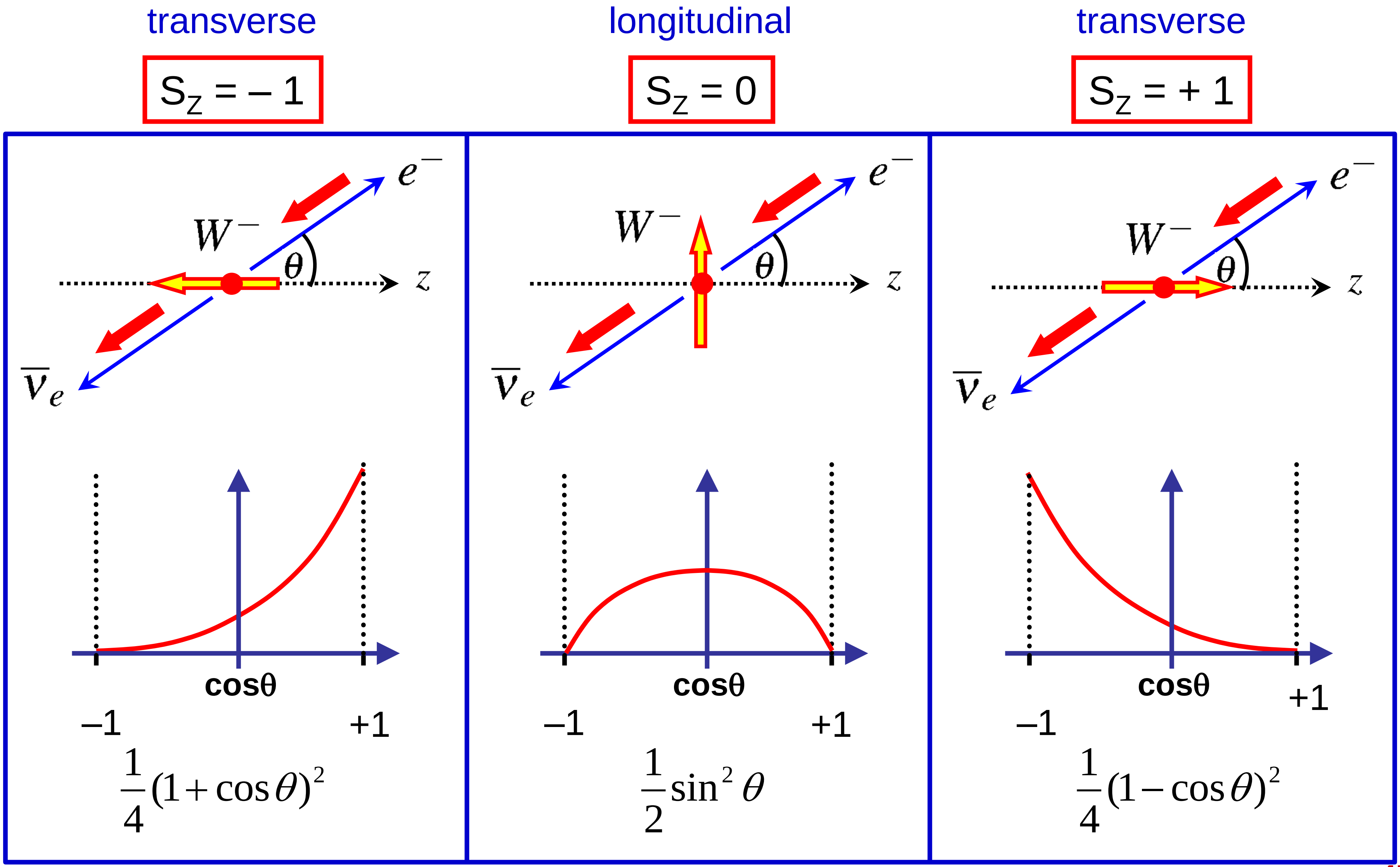
$$\varepsilon_L : \quad |M_L|^2 = \frac{1}{2}g^2M_W^2\sin^2\theta$$

$$\varepsilon_+ : \quad |M_+|^2 = \frac{1}{4}g^2M_W^2(1-\cos\theta)^2$$

Note $$|M_-|^2 + |M_L|^2 + |M_+|^2 = g^2M_W^2$$

For a sample of unpolarized W bosons, the decay is isotropic (as expected).

Angular Distributions



W Decay Rate

The decay rate is obtained using Fermi's golden rule

$$\frac{d\Gamma}{d\Omega} = \frac{p^*}{32\pi^2 M_W^2}|M|^2$$

with p* = $M_W/2$

$$\frac{d\Gamma_+}{d\Omega} = \frac{g^2 M_W}{64\pi^2}\frac{1}{4}(1+\cos\theta)^2 \qquad \frac{d\Gamma_L}{d\Omega} = \frac{g^2 M_W}{64\pi^2}\frac{1}{2}\sin^2\theta \qquad \frac{d\Gamma_-}{d\Omega} = \frac{g^2 M_W}{64\pi^2}\frac{1}{4}(1-\cos\theta)^2$$

and after integration over the solid angle $d\Omega = d\cos\theta\, d\phi$ one finds that the three polarization decay rates are identical

$$\Gamma_- = \Gamma_L = \Gamma_+ = \frac{g^2 M_W}{48\pi}$$

as one would expect since the decay rate cannot depend on the arbitrary definition of the z-axis.

For a sample of unpolarized W bosons, the decay is isotropic since each polarization state is equally likely: sum over all possible matrix elements and average over the three initial polarization states (i.e. $3 \times 1/3 = 1$):

$$\Gamma(W^- \to e^-\overline{\nu}) = \frac{g^2 M_W}{48\pi} = 230 \text{ MeV}$$

exp.t: 223 MeV

For Next Week

Study the material and prepare / ask questions
Study ch. read ch. 13 (sec. 1 to 7) in Halzen & Martin
and / or ch. 15, ch. 16, and app D in Thomson

Do the homeworks

Next week we will study the Higgs Mechanism
have a first look at the lecture notes, you can already have questions
read ch. 14 (sec. 5 to 9) and ch. 15 (sec. 1 to 6) in Halzen & Martin
and / or ch. 17 in Thomson